

Branes, black holes and topological strings on toric Calabi-Yau manifolds

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ABSTRACT: We develop means of computing exact degeneracies of BPS black holes on toric Calabi-Yau manifolds. We show that the gauge theory on the D4 branes wrapping ample divisors reduces to 2D q-deformed Yang-Mills theory on necklaces of \mathbf{P}^1 's. As explicit examples we consider local \mathbf{P}^2 , $\mathbf{P}^1 \times \mathbf{P}^1$ and A_k type ALE space times \mathbf{C} . At large N the D-brane partition function factorizes as a sum over squares of chiral blocks, the leading one of which is the topological closed string amplitude on the Calabi-Yau. This is in complete agreement with the recent conjecture of Ooguri, Strominger and Vafa.

KEYWORDS: Black Holes in String Theory, Topological Strings, p-branes.

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1. Introduction

Recently, Strominger, Ooguri and Vafa [1] made a remarkable conjecture relating four-dimensional BPS black holes in type II string theory compactified on a Calabi-Yau manifold X to the gas of topological strings on X . The conjecture states that the supersymmetric partition function Z_{brane} of the large number N of D-branes making up the black hole, is related to the topological string partition function Z^{top} as

$$Z_{brane} = |Z^{top}|^2,$$

to all orders in 't Hooft $1/N$ expansion. This provides an explicit proposal for what computes the corrections to the macroscopic Bekenstein-Hawking entropy of $d = 4$, $\mathcal{N} = 2$ black holes in type II string theory. Moreover, since the partition function Z_{brane} makes sense for any N , this is providing the non-perturbative completion of the topological string theory on X . A non-trivial test of the conjecture requires knowing topological string partition functions at higher genus on the one hand, and on the other explicit computation of D-brane partition functions. Since neither are known in general, some simplifying circumstances are needed.

Evidence that this conjecture holds was provided in [2][3] in a special class of local Calabi-Yau manifolds which are a neighborhood of a Riemann surface Σ . The conjecture for black holes preserving 4 supercharges was also tested to leading order in [4][5][6]. The conjecture was found to have extensions to 1/2 BPS black holes in compactifications with $\mathcal{N} = 4$ supersymmetry [7][8][4][5]. In [9] the version of the conjecture for open topological strings was formulated.

In this paper we consider black holes on local Calabi-Yau manifolds with torus symmetries. These Calabi-Yau manifolds are special in that the topological string amplitudes on them are computable to all genera. This makes them an ideal ground for testing the OSV conjecture. Namely, one of the most important aspects of the conjecture is that it provides a prediction for what the quantum corrections to the Bekenstein-Hawking entropy-area relation are. These invariably enter at higher (topological) string loops, and in this class of models they are computable. Moreover the Calabi-Yau manifolds end up involving local *surfaces* and this turns out to mean that the corresponding topological string amplitudes are far more complex than in the local curve cases studied in [2][3].

In the case of local curves in [2][3] the D-brane theory involved D4 branes wrapping a 4-cycle in the Calabi-Yau which was a line bundle over the Riemann surface Σ . The theory on N D4 branes turned out to be equivalent to a variant of the bosonic 2d $U(N)$ Yang-Mills theory on Σ . The amplitudes of this theory are computable as well, and this allowed an explicit verification of the OSV conjecture in that case. For the case at hand, it is natural to consider D4 branes wrapping 4-cycles that are invariant under torus symmetries of the Calabi-Yau. As long as one wants to preserve the symmetry, a generic such 4-cycle is a reducible sum of basic invariant divisors. The latter can be compact or not. The non-compact ones are of the form a line bundle over a Riemann surface, which is in this case necessarily a \mathbf{P}^1 – these are precisely the theories studied in [3]. The novelty here is that one typically gets more than one such divisor, so one has to consider intersecting

D4 branes. Like in [3] the theory on D4 branes of this kind is equivalent to the 2d qYM theory on the corresponding Riemann surface, in this case, a \mathbf{P}^1 . Since there is more than one divisor typically, what one ends up getting is a 2d qYM theory on chains of \mathbf{P}^1 's touching at points. The torus symmetries that rotate the \mathbf{P}^1 's around their equators require them to intersect over their north and south poles only. The rank of the gauge group can be different on different \mathbf{P}^1 's. So the new ingredient relative to [3] is understanding of what happens at the intersections. Clearly, at the intersections, we should have to insert certain point observables in the quantum Yang-Mills theory, that arise from integrating out bifundamental matter where the D4 branes intersect. We'll argue in section 3 that the contribution of these can be determined by a consistency argument.

The physics of D4 branes wrapping compact 4-cycles is a much harder mathematical problem without solution at present time. For example, in the case of local \mathbf{P}^2 , we would need to know the euler characters on moduli spaces of $SU(N)$ instantons on \mathbf{P}^2 , and to our knowledge, these are known only for $N = 2$ [10]. Here however, we need large N . Fortunately, understanding these does not turn out to be relevant for the black hole problem, as we now explain.

Namely, not all the D4 branes are expected to give rise to large black holes in four dimensions. This is true even in the compact case: Only the D4 branes wrapping 4-cycles that are the so called “very ample” divisors give rise to black holes with macroscopic horizons. In the present context, non-compactness of the irreducible divisor is a necessary condition for it to be ample. Moreover, because the Calabi-Yau manifold is non-compact, the 4-dimensional planck scale is going to infinity. Only by considering D4-branes which are also non-compact as in [2][3], one can keep the entropy of the black hole finite. So, fortune is with us, and in all cases where we expect the D-branes on these manifolds to be dual to black holes, the D-brane theory is computable.

There is one further subtlety. Because the D-branes are noncompact, different choices of boundary conditions at infinity on the branes give rise to different theories. Moreover, fixing the boundary conditions at infinity completely eliminates the moduli space of the holomorphic cycles that the D4 branes wrap. In the present setting, this means that topological string amplitudes that receive contributions from holomorphic maps to cycles that are far removed from the D4 brane, will have no relevance in describing the physics of the D-branes. Turning this around, a given D4 brane theory cannot be dual to topological strings on all of X , but only to the topological string on the local neighborhood of the D-brane in X . This further constrains the class of models that can have non-perturbative completion in terms of D4 branes and no D6 branes, but includes examples such as neighborhood of a shrinking \mathbf{P}^2 or $\mathbf{P}^1 \times \mathbf{P}^1$ in X , which we will study in this paper.

The paper has the following organization. In section 2 we review the conjecture of [1] focusing in particular to certain subtleties that are specific to the non-compact Calabi-Yau manifolds. We describe brane configurations which should be dual to topological strings on the Calabi-Yau. In section 3 we explain how to compute the corresponding partition functions Z_{brane} . The D4 brane theory turns out to be described by qYM theory on necklaces and chains of \mathbf{P}^1 . Where the different \mathbf{P}^1 's intersect, one gets insertions of certain observables corresponding to integrating out bifundamental matter from the

intersecting D4 branes. The qYM theory is solvable, and corresponding amplitudes can be computed exactly. In section 4 we present our first example of local \mathbf{P}^2 . We show that the 't Hooft large N expansion of the D-brane amplitude is related to the topological strings on the Calabi-Yau, and moreover and show that the version of the conjecture of [1] that is natural for non-compact Calabi-Yau manifolds [3] is upheld. In section 5 we consider an example of local $\mathbf{P}^1 \times \mathbf{P}^1$. In section 6 we consider N D-branes on (a neighborhood of an) A_k type ALE space. We show that at finite N our results coincide with that of H. Nakajima for Euler characteristics of moduli spaces of $U(N)$ instantons on ALE spaces, while in the large N limit we find precise agreement with the conjecture of [1].

2. Black holes on Calabi-Yau manifolds

Consider IIA string theory compactified on a Calabi-Yau manifold X . The effective $d = 4$, $\mathcal{N} = 2$ supersymmetric theory has BPS particles from D-branes wrapping holomorphic cycles in X . We will turn off the D6 brane charge, and consider arbitrary D0, D2 and D4 brane charges.

2.1 D-brane theory

Pick a basis of 2-cycles $[C^a] \in H_2(X, Z)$, and a dual basis of 4-cycles $[D_a] \in H_4(X, Z)$, $a = 1, \dots, h^{1,1}(X)$,

$$\#(D_a \cap C^b) = \delta_a^b.$$

This determines a basis for $h^{1,1} U(1)$ vector fields in four dimensions, obtained by integrating the RR 3-form C_3 on the 2-cycles C^a . Under these $U(1)$'s D2 branes in class $[C] \in H_2(X, Z)$ and D4 branes in class $[D] \in H_4(X, Z)$ carry electric and magnetic charges Q_{2a} and Q_4^b respectively:

$$[C] = \sum_a Q_{2a} [C^a], \quad [D] = \sum_a Q_4^a [D_a],$$

We also specify the D0 brane charge Q_0 . This couples to the one extra $U(1)$ vector multiplet which originates from RR 1-form.

The indexed degeneracy

$$\Omega(Q_4^a, Q_{2a}, Q_0)$$

of BPS particles in spacetime with charges Q_0, Q_{2a}, Q_4^a can be computed by counting BPS states in the Yang-Mills theory on the D4 brane [11]. This is computed by the supersymmetric path integral of the four dimensional theory on D in the topological sector with

$$Q_0 = \frac{1}{8\pi^2} \int_D \text{tr} F \wedge F, \quad Q_{2a} = \frac{1}{2\pi} \int_{C^a} \text{tr} F.$$

Since D is curved, this theory is topologically twisted, in fact it is the Vafa-Witten twist of the maximally supersymmetric $\mathcal{N} = 4$ theory on D .

2.2 Gravity theory

When the corresponding supergravity solution exists, the massive BPS particles are black holes in 4 dimensions, with horizon area given in terms of the charges

$$A_{BH} = \sqrt{\frac{1}{3!} C_{abc} Q_4^a Q_4^b Q_4^c} |Q'_0|$$

where C_{abc} are the triple intersection numbers of X , and $Q'_0 = Q_0 - \frac{1}{2} C^{ab} Q_{2a} Q_{2b}$.¹ The Bekenstein-Hawking formula relates this to the entropy of the black hole

$$S_{BH} = \frac{1}{4} A_{BH}.$$

For large charges, the macroscopic entropy defined by area, was shown to agree with the microscopic one [11][12]. The corrections to the entropy-area relation should be suppressed by powers in $1/A_{BH}$ (measured in plank units).

Following [13], Ooguri, Strominger and Vafa conjectured that, just as the leading order microscopic entropy can be computed by the classical area of the horizon and genus zero free energy F_0 of A-model topological string on X , the string loop corrections to the macroscopic entropy can be computed from higher genus topological string on X :

$$Z_{YM}(Q_4^a, \varphi^a, \varphi^0) = |Z^{top}(t^a, g_s)|^2 \tag{2.1}$$

where

$$Z_{YM}(Q_4^a, \varphi^a, \varphi^0) = \sum_{Q_{2a}, Q_0} \Omega(Q_4^a, Q_{2a}, Q_0) \exp(-Q_0 \varphi^0 - Q_{2a} \varphi^a).$$

is the partition function of the $\mathcal{N} = 4$ topological Yang-Mills with insertion of

$$\exp\left(-\frac{\varphi^0}{8\pi^2} \int \text{tr} F \wedge F - \sum_a \frac{\varphi^a}{2\pi} \int \omega_a \wedge \text{tr} F\right) \tag{2.2}$$

where we sum over all topological sectors.² The Kahler moduli of Calabi-Yau,

$$t^a = \int_{C^a} k + iB$$

and the topological string coupling constant g_s are fixed by the attractor mechanism:

$$t^a = \left(\frac{1}{2} Q_4^a + i\varphi^a\right) g_s$$

$$g_s = 4\pi/\varphi^0$$

Moreover, since the loop corrections to the macroscopic entropy are suppressed by powers of $1/N^2$ where $N \sim (C_{abc} Q_4^a Q_4^b Q_4^c)^{1/3}$ [12] the duality in (2.1) should be a large N duality in the Yang Mills theory.

¹ $C^{ab} C_{bd} = \delta_a^c, \quad C_{ab} = C_{abc} Q_4^c$
²Above, ω_a are dual to $C^a, \int_{C^a} \omega_b = \delta^a_b$.

2.3 D-branes for large black holes

Evidence that the conjecture (2.1) holds was provided in [2][3] for a very simple class of Calabi-Yau manifolds. We show in this paper that this extends to a broader class, provided that the classical area of the horizon is large. This imposes a constraint on the divisor D , which is what we turn to next.

Recall that for every divisor D on X there is a line bundle \mathcal{L} on X and a choice of a section s_D such that D is the locus where this section vanishes,

$$s_D = 0.$$

Different choices of the section correspond to homologous divisors on X , so the choice of $[D] \in H_4(X, \mathbf{Z})$ is the choice of the first Chern-class of \mathcal{L} (this is just Poincare duality but the present language will be somewhat more convenient for us) .

The classical entropy of the black hole is large when $[D]$ is deep inside the Kahler cone of X , [12] , i.e. $[D]$ is a “very ample divisor”. Then, intersection of $[D]$ with any 2-cycle class on X is positive, which guarantees that

$$C_{abc}t^a t^b t^c \gg 0.$$

Moreover, the attractor values of the Kahler moduli are also large and positive

$$Re(t^a) \gg 0.$$

Interestingly, this coincides with the case when the corresponding twisted $\mathcal{N} = 4$ theory is simple. Namely, the condition that $[D]$ is very ample is equivalent to

$$h^{2,0}(D) > 0.$$

When this holds, [14],[10] , the Vafa-Witten theory can be solved through mass deformation. In contrast, when this condition is violated, the twisted $\mathcal{N} = 4$ theory has lines of marginal stability, where BPS states jump, and background dependence.³

In the next subsection, we will give an example of a toric Calabi-Yau manifold with configurations of D4 branes satisfying the above condition.

2.4 An Example

Take X to be

$$X = O(-3) \rightarrow \mathbf{P}^2.$$

This is a toric Calabi-Yau which has a $d = 2$ $\mathcal{N} = (2, 2)$ linear sigma model description in terms of one $U(1)$ vector multiplet and 4 chiral fields X_i , $i = 0, \dots, 3$ with charges $(-3, 1, 1, 1)$. The Calabi-Yau X is the Higgs branch of this theory obtained by setting the D-term potential to zero,

$$|X_1|^2 + |X_2|^2 + |X_3|^2 = 3|X_0|^2 + r_t$$

³We thank C. Vafa for discussions which led to the statements here.

and modding out by the $U(1)$ gauge symmetry. The Calabi-Yau is fibered by T^3 tori, corresponding to phases of the four X 's modulo $U(1)$. Above, $r_t > 0$ is the Kahler modulus of X , the real part of $t = \int_{C_t} k + iB$. The Kahler class $[k]$ is a multiple of the integral class $[D_t]$ which generates $H^2(X, \mathbb{Z})$, $[k] = r_t [D_t]$.

Consider now divisors on X . A divisor in class

$$[D] = Q [D_t]$$

is given by zero locus of a homogenous polynomial in X_i of charge Q in the linear sigma model:

$$D : \quad s_D^Q(X_0, \dots, X_3) = 0.$$

In fact s_D^Q is a section of a line bundle over X of degree $Q[D_t]$. A generic such divisor breaks the $U(1)^3$ symmetry of X which comes from rotating the T^3 fibers. There are special divisors which preserve these symmetries, obtained by setting X_i to zero,

$$D_i : \quad X_i = 0.$$

It follows that $[D_{1,2,3}] = [D_t]$, and that $[D_0] = -3[D_t]$. The divisor D_0 corresponds to the \mathbf{P}^2 itself, which is the only compact holomorphic cycle in X .

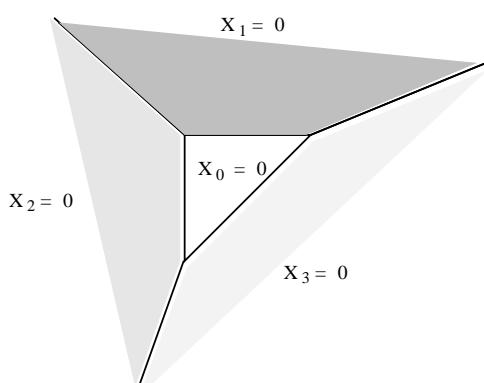


Figure 1: Local \mathbf{P}^2 . We depicted the base of the T^3 fibration which is the interior of the convex polygon in \mathbf{R}^3 . The shaded planes are its faces.

As explained above, we are interested in D4 branes wrapping divisors whose class $[D]$ is positive, $Q = Q_4 > 0$. Since the compact divisors have negative classes, any divisor in this class is non-compact 4-cycle in X . The divisors have a moduli space \mathcal{M}_Q , the moduli space of charge Q polynomials, which is very large in this case since X is non-compact and the linear sigma model contains a field X_0 of negative degree. If D were compact, the theory on the D4 brane would involve a sigma model on \mathcal{M}_Q . Since D is not compact, in formulating the D4 brane theory we have to pick boundary conditions at infinity. This picks a point in the moduli space \mathcal{M}_Q , which is a particular divisor D .

Now, consider the theory on the D4 brane on D . Away from the boundaries of the moduli space \mathcal{M}_Q , the theory on the D4 brane should not depend on the choice of the divisor, but only on the topology of D . In the interior of the moduli space, D intersects the \mathbf{P}^2 along a curve Σ of degree Q , which is generically an irreducible and smooth curve of genus $g = (Q-1)(Q-2)/2$, and D is a line bundle over it. The theory on the brane is a Vafa-Witten twist of the maximally supersymmetric $\mathcal{N} = 4$ gauge theory with gauge group $G = U(1)$. At the boundaries of the moduli space, Σ and D can become reducible. For example, Σ can collapse to a genus zero, degree Q curve by having $s^Q = X_1^Q$, corresponding to having $D = Q \cdot D_1$. Then D is an $O(-3)$ bundle over \mathbf{P}^1 , and the theory on the D4

brane wrapping D is the twisted $\mathcal{N} = 4$ theory with gauge group $G = U(Q)$ with scalars valued in the normal bundle to D .

Both of these theories were studied recently in [3] in precisely this context. In both cases, the theory on the D4 brane computes the numbers of BPS bound-states of D0 and D2 brane with the D4 brane. Correspondingly, the topological string which is dual to this in the $1/Q$ expansion describes *only* the maps to X which fall in the neighborhood of D . In other words, the D4 brane theory is computing the non-perturbative completion of the topological string on X_D where X_D is the total space of the normal bundle to D in X . It is not surprising that the YM theory on the (topologically) distinct divisors D gives rise to different topological string theories – because D is non-compact, different choices of the boundary conditions on D give rise to a-priori different QFTs.

It is natural to ask if there is a choice of the divisor D for which we can expect the YM theory to be dual to the topological string on $X = \mathcal{O}(-3) \rightarrow \mathbf{P}^2$. Consider a toric divisor in the class $[D] = Q[D_t]$ of the form

$$D = N_1 D_1 + N_2 D_2 + N_3 D_3 \tag{2.3}$$

where $Q = N_1 + N_2 + N_3$ for N_i positive integers. The D4 brane on D will form bound-states with D2 branes running around the edges of the toric base, and arbitrary number of the D0 branes. Recall furthermore that, because X has $U(1)$ symmetries, the topological string on X localizes to maps fixed under the torus actions, i.e. maps that in the base of the Calabi-Yau project to the edges. It is now clear that the D4 branes on D in (2.3) are the natural candidate to give the non-perturbative completion of the topological string on X . We will see in the next sections that this expectation is indeed fully realized.

The considerations of this section suggest that of all the toric Calabi-Yau manifolds, only a few are expected to have non-perturbative completions in terms of D4 branes. The necessary condition translates into having at most one compact 4-cycle in X , so that the topological string on the neighborhood X_D of an ample divisor can agree with the topological string on all of X . Even so, the available examples have highly non-trivial topological string amplitudes, providing a strong test of the conjecture.

3. The D-brane partition function

In the previous section we explained that D4-branes wrapping non-compact, toric divisors should be dual to topological strings on the toric Calabi-Yau threefold X . The divisor D in question are invariant under T^3 action on X , and moreover generically reducible, as the local \mathbf{P}^2 case exemplifies. In this section we want to understand what is the theory on the D4 brane wrapping D .

Consider the local \mathbf{P}^2 with divisor D as in (2.3). Since D is reducible, the theory on the branes is a topological $\mathcal{N} = 4$ Yang-Mills with quiver gauge group $G = U(N_1) \times U(N_2) \times U(N_3)$. The topology of each of the three irreducible components is

$$D_i : \quad \mathcal{O}(-3) \rightarrow \mathbf{P}^1$$

In the presence of more than one divisor, there will be additional bifundamental hypermultiplets localized along the intersections. Here, D_1 , D_2 and D_3 intersect pairwise along three copies of a complex plane at $X_i = 0 = X_j$, $i \neq j$.

As shown in [2][3], the four-dimensional twisted $\mathcal{N} = 4$ gauge theory on

$$\mathcal{O}(-p) \rightarrow \mathbf{P}^1$$

with (2.2) inserted is equivalent to a cousin of two dimensional Yang-Mills theory on the base $\Sigma = \mathbf{P}^1$ with the action

$$S = \frac{1}{g_s} \int_{\Sigma} \text{tr} \Phi \wedge F + \frac{\theta}{g_s} \int_{\Sigma} \text{tr} \Phi \wedge \omega_{\Sigma} - \frac{p}{2g_s} \int_{\Sigma} \text{tr} \Phi^2 \wedge \omega_{\Sigma} \quad (3.1)$$

where $\theta = \varphi^1/2\pi g_s$. The four dimensional theory localizes to constant configurations along the fiber. The field $\Phi(z)$ comes from the holonomy of the gauge field around the circle at infinity:

$$\int_{\text{fiber}} F(z) = \oint_{S^1_{z,\infty}} A(z) = \Phi(z). \quad (3.2)$$

Here the first integral is over the fiber above a point on the base Riemann surface with coordinate z . The (3.1) is the action, in the Hamiltonian form, of a $2d$ YM theory, where

$$\Phi(z) = g_s \frac{\partial}{\partial A(z)}$$

is the momentum conjugate to A . However, the theory is *not* the ordinary YM theory in two dimensions. This is because the the field Φ is periodic. It is periodic since it comes from the holonomy of the gauge field at infinity. This affects the measure of the path integral for Φ is such that not Φ but $\exp(i\Phi)$ is a good variable. The effect of this is that the theory is a deformation of the ordinary YM theory, the ‘‘quantum’’ YM theory [3].

Integrating out the bifundamental matter fields on the intersection should, from the two dimensional perspective, correspond to inserting point observables where the \mathbf{P}^1 's meet in the \mathbf{P}^2 base. We will argue in the following subsections, that the point observable corresponds to

$$\sum_{\mathcal{R}} \text{Tr}_{\mathcal{R}} V_{(i)}^{-1} \text{Tr}_{\mathcal{R}} V_{(i+1)} \quad (3.3)$$

where

$$V_{(i)} = e^{i\Phi^{(i)} - i\oint A^{(i)}}, \quad V_{(i+1)} = e^{i\Phi^{(i+1)}}$$

The point observables $\Phi^{(i)}$ and $\Phi^{(i+1)}$ are inserted where the \mathbf{P}^1 's intersect, and the integral is around a small loop on \mathbf{P}^1_i around the intersection point. The sum is over all representations \mathcal{R} that exist as representations of the gauge groups on both \mathbf{P}^1_i and \mathbf{P}^1_{i+1} . This means effectively one sums over the representations of the gauge group of smaller rank.

By topological invariance of the YM theory, the interaction (3.3) depends only on the geometry near the intersections of the divisors, and not on the global topology. For intersecting non-compact toric divisors, this is universal, independent of either D or X . In the following subsection we will derive this result.

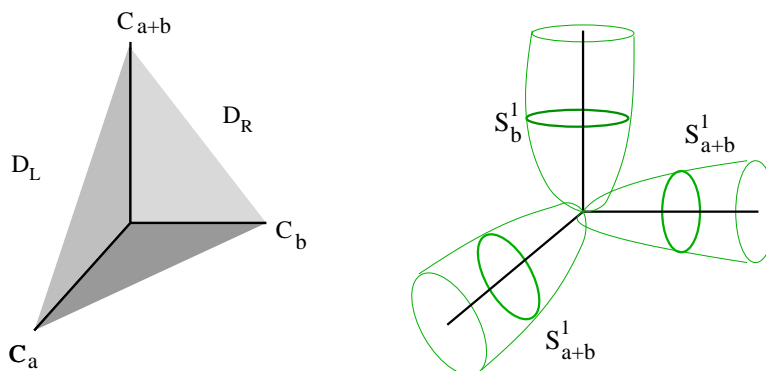


Figure 2: D4-branes are wrapped on the divisors $D_{L,R} = \mathbf{C}^2$. The three boldfaced lines in the figure on the left correspond to three disks C_a, C_b, C_{a+b} over which the a, b and $(a+b)$ 1-cycles of the lagrangian $T^2 \times \mathbf{R}$ fibration degenerate. The cycles of the T^2 which are finite are depicted in the figure on right.

3.1 Intersecting D4 branes

In this subsection we will motivate the interaction (3.3) between D4-branes on intersecting divisors. The interaction between the D4 branes comes from the bifundamental matter at the intersection and, as explained above, since the matter is localized and the theory topological, integrating it out should correspond to universal contributions to path integral over D_L and D_R that are independent of the global geometry. Therefore, we might as well take D 's, and X itself to be particularly simple, and the simplest choice is two copies of the complex 2-plane \mathbf{C}^2 in $X = \mathbf{C}^3$. We can think of the pair of divisors as line bundles fibered over disks C_a and C_b . One might worry that something is lost by replacing Σ by a non-compact Riemann surface, but this is not the case – as was explained in [3] because the theory is topological, we can reconstruct the theory on any X from simple basic pieces by gluing, and what we have at hand is precisely one of these building blocks.

The fields at the intersection $C_{a+b} = D_L \cap D_R$ transform in the bifundamental (M, \bar{N}) representations of the $U(M) \times U(N)$ gauge groups on the D-branes. We will first argue that the effect of integrating them out is insertion of

$$\sum_{\mathcal{R}} \text{Tr}_{\mathcal{R}} \exp(i \oint_{S_b^1} A^{(L)}) \text{Tr}_{\mathcal{R}} \exp(i \oint_{S_b^1} A^{(R)}) \quad (3.4)$$

where $\oint_{S_b^1} A^{(L)}$ and $\oint_{S_b^1} A^{(R)}$ are the holonomies of the gauge fields on D_L and D_R respectively around the circle at infinity on the cap C_{a+b} , i.e. $S_b^1 = \partial C_{a+b}$, see figure 2. (If this notation seems odd, it will stop being so shortly).

We will argue this by consistency as follows.⁴ First, note that there is correlation between turning of certain fluxes on D_L and D_R . To see this note that, if one adds D2 branes along C_{a+b} , the D2 branes have the effect of turning on flux on *both* D_L and D_R . Consider for simplicity the case where $M = 1 = N$. The fact that the corresponding fluxes are correlated is the statement that $\int F^{(L)} = \int F^{(R)}$ where integrals are taken over the

⁴We thank C. Vafa for suggesting use of this approach.

fibers over a point on C_{a+b} in the divisors D_L and D_R respectively, where we view $D_{L,R}$ as fibrations over C_{a+b} . Since $S_b^1 = \partial C_{a+b}$ this is equivalent to

$$\oint_{S_{a+b}^1} A^{(L)} = \oint_{S_{a+b}^1} A^{(R)} \tag{3.5}$$

where S_{a+b}^1 is the one cycle in X that *vanishes* over C_{a+b} (this cycle is well defined in X as we will review shortly). This is consistent with insertion of

$$\sum_{n \in \mathbf{Z}} \exp(in \oint_{S_b^1} A^{(L)}) \exp(in \oint_{S_b^1} A^{(R)}). \tag{3.6}$$

because $\oint_{S_b^1} A^{(L,R)}$ and $\oint_{S_{a+b}^1} A^{(L,R)}$ are canonically conjugate, (one way to see this is to consider the qYM theory one gets on C_{a+b} . Then insertion of (3.6) implies (3.5) as an identity inside correlation functions). For general M, N gauge and Weyl invariance imply precisely (3.4) .

We must still translate the operators that appear in (3.4) , in terms of operators $\Phi^{(L,R)}$ and $A^{(L,R)}$ in the qYM theories on C_a and C_b . This requires understanding of certain aspects of T^3 fibrations. While any toric Calabi-Yau threefold is a lagrangian T^3 fibration, it is also a special lagrangian $T^2 \times \mathbf{R}$ fibration, where over each of the edges in the toric base a (p, q) cycle of the T^2 degenerates. The one-cycle which remains finite over the edge is ambiguous. In the case at hand, we have \mathbf{C}^3 with two D4 branes, and this provides a particularly natural choice. Namely, for cycle which is finite over C_a chose the cycle that vanishes over the other edge of D_L , and similarly for C_b and D_R . This makes the gluing rules particularly simple.⁵ This is described in figure 2. In the figure, the 1-cycles of the T^2 that vanish over C_a, C_b and C_{a+b} are S_a^1, S_b^1, S_{a+b}^1 , respectively. These determine the point observables Φ 's in the qYM theories on the corresponding disk. The 1-cycles that remain finite are S_{a+b}^1, S_b^1 and S_{a+b}^1 , in order. It follows that

$$\oint_{S_b^1} A^{(L)} = \oint_{S_{a+b}^1} A^{(L)} - \Phi^{(L)}, \quad \oint_{S_b^1} A^{(R)} = \Phi^{(R)},$$

which justifies (3.3) . In the next subsection we will compute the qYM amplitudes with these observables inserted.

3.2 Partition functions of qYM

Like ordinary two dimensional YM theory, the qYM theory is solvable exactly [3] . In this subsection we will compute the YM partition functions with the insertions of observables (3.3) . In [3] it was shown that qYM partition function $Z(\Sigma)$ on an arbitrary Riemann surface Σ can be computed by means of operatorial approach. Since the theory is invariant under area preserving diffeomorphisms, knowing the amplitudes for Σ an annulus A , a pant P and a cap C , completely solves the theory – amplitudes on any Σ can be obtained from

⁵In the language on next subsection, this corresponds to inserting $q^{pC_2(\mathcal{R})}$ as a propagator to get $\mathcal{O}(-p)$ line bundle over \mathbf{P}^1 .

this by gluing. In the present case, we will only need the cap and the annulus amplitudes, but with insertions of observables. Since the Riemann surfaces in question are embedded in a Calabi-Yau, we are effectively sewing Calabi-Yau manifolds, so one also has to keep track of the data of the fibration. The rules of gluing a Calabi-Yau manifold out of \mathbf{C}^3 patches are explained in [15] and we will only spell out their consequences in the language of 2d qYM.

In the previous subsection, the theory on divisors D_L and D_R in \mathbf{C}^3 was equivalent to qYM theories on disks C_a and C_b , with some observable insertions. These are Riemann surfaces with a boundary, so the corresponding path integrals define states in the Hilbert space of qYM theory on S^1 . Keeping the holonomy $U = Pe^{i\oint A}$ fixed on the boundary, the corresponding wave function can be expressed in terms of characters of irreducible representations \mathcal{R} of $U(N)$ as:

$$Z(U) = \sum_{\mathcal{R}} Z_{\mathcal{R}} \text{Tr}_{\mathcal{R}} U$$

The first thing we will answer is how to compute the corresponding states, and then we will see how to glue them together. As we saw in the previous section, the choice of the coordinate $\oint_{S^1} A$ on the boundary is ambiguous, as the choice of the cycle which remains finite is ambiguous. This ambiguity is related to the choice of the Chern class of a line bundle over a non-compact Riemann surface, i.e. how the divisors $D_{L,R}$ are fibered over the corresponding disks. The simplest choice is the one that gives trivial fibration, and this is the one we made in figure 2 (this corresponds to picking the cycle that vanishes over C_{a+b}).

The partition function on a disk with trivial bundle over it and no insertions is

$$Z(C)(U) = \sum_{\mathcal{R} \in U(N)} S_{0\mathcal{R}} e^{i\theta C_1(\mathcal{R})} \text{Tr}_{\mathcal{R}} U, \tag{3.7}$$

Above, $C_1(\mathcal{R})$ is the first casimir of the representation \mathcal{R} , and $S_{\mathcal{R}\mathcal{P}}(N, g_s)$ is a relative of the S-matrix of the $U(N)$ WZW model

$$S_{\mathcal{R}\mathcal{Q}}(N, g_s) = \sum_{w \in S_N} \epsilon(w) q^{-(\mathcal{R} + \rho_N) \cdot w(\mathcal{Q} + \rho_N)}, \tag{3.8}$$

where

$$q = \exp(-g_s)$$

and S_N is Weyl group of $U(N)$ and ρ_N is the Weyl vector.⁶

Sewing Σ_L and Σ_R is done by

$$Z(\Sigma_L \cup \Sigma_R) = \int dU Z(\Sigma_L)(U) Z(\Sigma_R)(U^{-1}) = \sum_{\mathcal{R}} Z_{\mathcal{R}}(\Sigma_L) Z_{\mathcal{R}}(\Sigma_R)$$

⁶The normalization of the path integral is ambiguous. In our examples in sections 4-6 we will choose it in such a way that the amplitudes agree with the topological string in the large N limit.

For example, the amplitude corresponding to $\Sigma = \mathbf{P}^1$ with $O(-p)$ bundle over it and no insertions can be obtained by gluing two disks and an annulus with $O(-p)$ bundle over it:

$$Z(A, p)(U_1, U_2) = \sum_{\mathcal{R} \in U(N)} q^{pC_2(\mathcal{R})/2} e^{i\theta C_1(\mathcal{R})} \text{Tr}_{\mathcal{R}} U_1 \text{Tr}_{\mathcal{R}} U_2 \quad (3.9)$$

This gives

$$Z(P^1, p) = \sum_{\mathcal{R}} (S_{0\mathcal{R}})^2 q^{pC_2(\mathcal{R})/2} e^{i\theta C_1(\mathcal{R})} \quad (3.10)$$

In addition we will need to know how to compute expectation values of observables in this theory. As we will show in the appendix B, the amplitude on a cap with a trivial line bundle and observable $\text{Tr}_Q e^{i\Phi - in \oint_{S^1} A}$ inserted equals

$$Z(C, \text{Tr}_Q e^{i\Phi - in \oint_{S^1} A})(U) = \sum_{\mathcal{R}} q^{\frac{n}{2}C_2(\mathcal{Q})} S_{\mathcal{Q}\mathcal{R}}(N, g_s) \text{Tr}_{\mathcal{R}} U. \quad (3.11)$$

where U is the holonomy on the boundary.

It remains to compute the expectation value of the observables in (3.3) in the two-dimensional theory on C_a and C_b . The amplitude on the intersecting divisors D_L, D_R is

$$\begin{aligned} Z(V)(U^{(L)}, U^{(R)}) &= \sum_{\mathcal{Q} \in U(M), \mathcal{P} \in U(N)} V_{\mathcal{Q}\mathcal{P}}(M, N) \text{Tr}_{\mathcal{Q}} U^{(L)} \text{Tr}_{\mathcal{P}} U^{(R)} \\ V_{\mathcal{Q}\mathcal{P}}(M, N) &= \sum_{\mathcal{R} \in U(M)} S_{\mathcal{Q}\mathcal{R}}(M, g_s) q^{\frac{1}{2}C_2^{(M)}(\mathcal{R})} S_{\mathcal{R}\mathcal{P}}(N, g_s) \end{aligned} \quad (3.12)$$

In the above, $U^{(L,R)}$ is the holonomy at the boundary of C_a and C_b .

When $M = N$, there is a simpler expression for the vertex amplitude in (3.12). Using the definition of $S_{\mathcal{P}\mathcal{R}}$ (3.8) and summing over \mathcal{R} we have

$$V_{\mathcal{P}\mathcal{Q}} = \theta^N(q) q^{-\frac{1}{2}C_2(\mathcal{P})} S_{\overline{\mathcal{P}}\mathcal{Q}} q^{-\frac{1}{2}C_2(\mathcal{Q})} \quad (3.13)$$

and where $\theta(q) = \sum_{m \in \mathbf{Z}} q^{\frac{m^2}{2}}$. This is related to the familiar realization in WZW models of the relation

$$STS = (TST)^{-1}$$

between $SL(2, \mathbf{Z})$ generators S and T in WZW models where

$$T_{\mathcal{R}\mathcal{Q}} = q^{\frac{1}{2}C_2(\mathcal{R})} \delta_{\mathcal{R}\mathcal{Q}}, \quad S_{\mathcal{R}\mathcal{P}}^{-1}(g_s, N) = S_{\mathcal{R}\mathcal{P}}(-g_s, N) = S_{\overline{\mathcal{R}}\mathcal{P}}(g_s, N). \quad (3.14)$$

The difference is that there is no quantization of the level k here. Even at a non-integer level, this is more straightforward in the $SU(N)$ case, where the theta function in (3.13) would not have appeared.

3.3 Modular transformations

The partition functions of D4 branes on various divisors with chemical potentials

$$S_{4d} = \frac{1}{2g_s} \int \text{tr } F \wedge F + \frac{\theta}{g_s} \int \text{tr } F \wedge \omega,$$

turned on, are computing degeneracies of bound-states of Q_2 D2 branes and Q_0 D0 branes with the D4 branes, where

$$Q_0 = \frac{1}{8\pi^2} \int \text{tr } F \wedge F, \quad Q_2 = \frac{1}{2\pi} \int \text{tr } F \wedge \omega, \quad (3.15)$$

so the YM amplitudes should have an expansion of the form

$$Z^{q\text{YM}} = \sum_{q_0, q_1} \Omega(Q_0, Q_2, Q_4) \exp \left[-\frac{4\pi^2}{g_s} Q_0 - \frac{2\pi\theta}{g_s} Q_2 \right]. \quad (3.16)$$

The amplitudes we have given are not expansions in $\exp(-1/g_s)$, but rather in $\exp(-g_s)$, so the existence of the (3.16) expansion is not apparent at all. The underlying $\mathcal{N} = 4$ theory however has S duality that relates strong and weak coupling expansions, so we should be able to make contact with (3.16).

Since amplitudes on more complicated manifolds are obtained from the simpler ones by gluing, it will suffice for us to show this for the propagators, vertices and caps. Consider the annulus amplitude (3.9) Using the Weyl-denominator form of the $U(N)$ characters $\text{Tr}_{\mathcal{R}} U = \Delta_H(u)^{-1} \sum_{w \in S_N} (-)^w e^{\omega(iu) \cdot (\mathcal{R} + \rho_N)}$ we can rewrite $Z(A, p)$ as

$$Z(A, p)(U, V) = \Delta_H(u)^{-1} \Delta_H(v)^{-1} \sum_{n \in \mathbf{Z}^N} \sum_{w \in S_N} q^{\frac{n^2}{2}} e^{n(iu - w(iv))}$$

which is manifestly a modular form,⁷ which we can write

$$Z(A, p)(U, V) = \Delta_H(u)^{-1} \Delta_H(v)^{-1} \left(\frac{g_s p}{2\pi} \right)^{-\frac{N}{2}} \sum_{m \in \mathbf{Z}^N} \sum_{w \in S_N} \tilde{q}^{\frac{1}{2p}} \left(m - \frac{u - w(v)}{2\pi} \right)^2 \quad (3.17)$$

where in terms of $\tilde{q} = e^{-4\pi^2/g_s}$. In the above, the eigenvalues U_i of U are written as $U_i = \exp(iu_i)$, and $\Delta_H(u)$ enters the Haar measure:

$$\int dU = \int \prod_i du_i \Delta_H(u)^2$$

Note that, in gluing, the determinant $\Delta_H(u)^2$ factors cancel out, and simple degeneracies will be left over.

Similarly, the vertex amplitude (3.12) corresponding to intersection of N and M D4 branes can be written as (see appendix C for details):

$$\begin{aligned} Z(U, V) &= \Delta_H(u)^{-1} \Delta_H(v)^{-1} \theta^M(q) \sum_{m \in \mathbf{Z}^M} q^{-\frac{1}{2}m^2} e^{m \cdot v} \\ &\times \sum_{w \in S_N} (-)^w \sum_{n \in \mathbf{Z}^N} e^{n \cdot (w(iu) + iv - g_s(\rho_N - \rho_M))} \end{aligned} \quad (3.18)$$

⁷Recall, $\theta(\tau, u) = (-i\tau)^{-\frac{1}{2}} e^{-i\pi \frac{u^2}{\tau}} \theta\left(-\frac{1}{\tau}, \frac{u}{\tau}\right)$, where $\theta(\tau, u) = \sum_{n \in \mathbf{Z}} e^{i\tau n^2} e^{2\pi i u n}$.

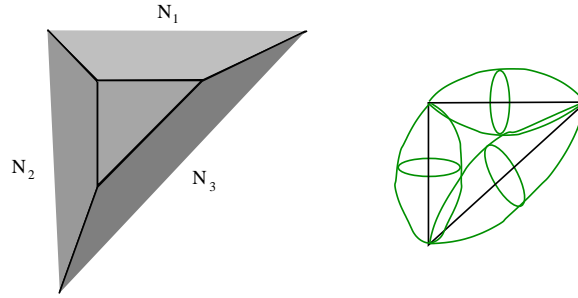


Figure 3: Local \mathbf{P}^2 , depicted as a toric web diagram. The numbers of D4 branes wrapping the torus invariant non-compact 4-cycles are specified.

where v, ρ_M are regarded as N dimensional vectors, the last $N - M$ of whose entries are zero. We see that $Z(U, V)$ is given in terms of theta functions, so it is modular form, its modular transform given by

$$\begin{aligned}
 Z(U, V) &= \Delta_H(u)^{-1} \Delta_H(v)^{-1} \left(\frac{g_s}{2\pi} \right)^{-M/2} \theta^M(\tilde{q}) \\
 &\sum_{m \in \mathbf{Z}^M} \tilde{q}^{-\frac{1}{2}(m+iv/2\pi)^2} \\
 &\sum_{w \in S_N} (-)^w \sum_{n \in \mathbf{Z}^N} e^{n(\cdot w(iu)+iv-g_s(\rho_N-\rho_M))}
 \end{aligned} \tag{3.19}$$

In a given problem, it is often easier to compute the degeneracies of the BPS states from the amplitude as a whole, rather than from the gluing the S-dual amplitudes as in (3.19). Nevertheless, modularity at the level of vertices, propagators and caps, demonstrates that the $1/g_s$ expansion of our amplitudes does exist in a general case.

4. Branes and black holes on local \mathbf{P}^2 .

We will now use the results of the previous section to study black holes on $X = O(-3) \rightarrow \mathbf{P}^2$. As explained in section 2, to get large black holes on $\mathbf{R}^{3,1}$ we need to consider D4 branes wrapping very-ample divisors on X , which are then necessarily non-compact. Moreover, the choice of divisor D that should give rise to a dual of topological strings on X corresponds to

$$D = N_1 D_1 + N_2 D_2 + N_3 D_3$$

where D_i , $i = 1, 2, 3$ are the toric divisors of section 2.

Using the results of section 3, it is easy to compute the amplitudes corresponding to the brane configuration. We have $N_1 \geq N_2 \geq N_3$ D4 branes on three divisors of topology $D_i = O(-3) \rightarrow \mathbf{P}^1$. From each, we get a copy of quantum Yang Mills theory on \mathbf{P}^1 with $p = 3$, as discussed in section 3. From the matter at the intersections, we get in addition, insertion of observables (3.3) at two points in each \mathbf{P}^1 .

All together this gives:

$$Z_{qYM} = \alpha \sum_{\mathcal{R}_i \in U(N_i)} V_{\mathcal{R}_2 \mathcal{R}_1}(N_2, N_1) V_{\mathcal{R}_3 \mathcal{R}_2}(N_3, N_2) V_{\mathcal{R}_3 \mathcal{R}_1}(N_3, N_1) \prod_{j=1}^3 q^{\frac{3C_2(\mathcal{R}_i)}{2}} e^{i\theta_i C_1(\mathcal{R}_i)} \quad (4.1)$$

Note that in the physical theory there should be only one chemical potential for D2-branes, corresponding to the fact that $H_2(X, \mathbf{Z})$ is one dimensional. In the theory of the D4 brane we $H_2(D, \mathbf{Z})$ is three dimensional, generated by the 3 \mathbf{P}^1 's in D – the three chemical potentials θ_i above couple to the D2 branes wrapping these. While all of these D2 branes should correspond to BPS states in the Yang-Mills theory, not all of them should correspond to BPS states once the theory is embedded in the string theory. Because the three \mathbf{P}^1 's that the D2 brane wrap are all homologous in $H_2(X, \mathbf{Z})$,

$$[\mathbf{P}_1^1] - [\mathbf{P}_3^1] \sim 0, \quad [\mathbf{P}_2^1] - [\mathbf{P}_3^1] \sim 0$$

there will be D2 brane instantons that can cause those BPS states that carry charges in $H_2(D, \mathbf{Z})$ to pair up into long multiplets. Decomposing $H_2(D, \mathbf{Z})$ into a $H_2(D, \mathbf{Z})^{\parallel} = H_2(X, \mathbf{Z})$ and $H_2(D, \mathbf{Z})^{\perp}$, it is natural to turn off the the chemical potentials for states with charges in $H_2(D, \mathbf{Z})^{\perp}$. This corresponds to putting

$$\theta_i = \theta, \quad i = 1, 2, 3.$$

For some part, we will keep the θ -angles different, but there is only one θ natural in the theory.

The normalization α of the path integral is chosen in such a way that Z_{qYM} has chiral/anti-chiral factorization in the large N_i limit (see 4.6 and 4.10 below).

$$\alpha = q^{-\left(\rho_{N_2}^2 + \frac{N_2}{24}\right)} q^{-2\left(\rho_{N_3}^2 + \frac{N_3}{24}\right)} e^{\frac{(N_1+N_2+N_3)\theta^2}{6g_s}} q^{\frac{(N_1+N_2+N_3)^3}{72}}$$

The partition function simplifies significantly if we take equal numbers of the D4 branes on each D_i ,

$$N_i = N, \quad i = 1, 2, 3$$

since in this case, we can replace (3.12) form of the vertex amplitude with the simpler (3.13) , and the D-brane partition function becomes

$$Z_{qYM} = \alpha \theta^{3N}(q) \sum_{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \in U(N)} S_{\mathcal{R}_1 \bar{\mathcal{R}}_2}(g_s, N) S_{\mathcal{R}_2 \bar{\mathcal{R}}_3}(g_s, N) S_{\mathcal{R}_3 \bar{\mathcal{R}}_1}(g_s, N) \times \prod_{j=1}^3 q^{\frac{C_2(\mathcal{R}_i)}{2}} e^{i\theta_i C_1(\mathcal{R}_i)} \quad (4.2)$$

In the following subsections we will first take the large N_i limit of Z_{qYM} to get the closed string dual of the system. We will then use modular properties of the partition function to compute the degeneracies of the BPS states of D0-D2-D4 branes.

4.1 Black holes from local \mathbf{P}^2

According to the conjecture of [1] (or more precisely, its version for the non-compact Calabi-Yau manifolds proposed in [3]) the large N limit of the D-brane partition function Z_{brane} , which in our case equals Z_{qYM} , should be given by

$$Z_{qYM}(D, g_s, \theta) \approx \sum_{\alpha} |Z_{\alpha}^{top}(t, g_s)|^2$$

where

$$t = \frac{1}{2}(N_1 + N_2 + N_3)g_s - i\theta$$

since $[D] = (N_1 + N_2 + N_3)[D_t]$ where $[D_t]$ is dual to the class that generates $H_2(X, \mathbf{Z})$. In the above, the two expressions should equal up to terms of order $O(\exp(-1/g_s))$, hence the “approximate” sign. The sum over α is the sum over chiral blocks which should correspond to the boundary conditions at infinity of X . More precisely, the leading chiral block should correspond to including only the normalizable modes of topological string on X , which count holomorphic maps to \mathbf{P}^2 , the higher ones containing fluctuation in the normal direction [3][9]. We will see below that this prediction is realized precisely.

The Hilbert space of the qYM theory, spanned by states labeled by representations \mathcal{R} of $U(N)$, at large N splits into

$$\mathcal{H}^{qYM} \approx \oplus_{\ell} \mathcal{H}_{\ell}^{+} \otimes \mathcal{H}_{\ell}^{-}$$

where \mathcal{H}_{ℓ}^{+} and \mathcal{H}_{ℓ}^{-} are spanned by representations R_{+} and R_{-} with small numbers of boxes as compared to N , and ℓ is the $U(1)$ charge. Correspondingly, the qYM partition function also splits as

$$Z_{qYM} \approx \sum_{\ell} Z_{\ell}^{+} Z_{\ell}^{-},$$

where Z_{ℓ}^{\pm} are the chiral and anti-chiral partitions. We will now compute these, and show that they are given by topological string amplitudes.

i. The $N_i = N$ case.

We’ll now compute the large N limit of the D-brane partition function (4.2) for $N_i = N$, $i = 1, 2, 3$. At large N , the $U(N)$ Casimirs in representation $\mathcal{R} = R_{+}\bar{R}_{-}[\ell_R]$ are given by

$$\begin{aligned} C_2(\mathcal{R}) &= \kappa_{R_{+}} + \kappa_{R_{-}} + N(|R_{+}| + |R_{-}|) + N\ell_R^2 + 2\ell_R(|R_{+}| - |R_{-}|), \\ C_1(\mathcal{R}) &= N\ell_R + |R_{+}| - |R_{-}| \end{aligned} \tag{4.3}$$

where

$$\kappa_R = \sum_{i=1}^{N-1} R_i(R_i - 2i + 1)$$

and $|R|$ is the number of boxes in R .

The S-matrix $S_{\mathcal{RQ}}$ is at large N given in [9]

$$\begin{aligned}
 q^{-(\rho^2 + \frac{N}{24})} S_{\mathcal{RQ}}(-g_s, N) &= M(q^{-1}) \eta(q^{-1})^N (-)^{|R_+|+|R_-|+|Q_+|+|Q_-|} \\
 &\times q^{N\ell_R \ell_Q} q^{\ell_Q(|R_+|-|R_-|)} q^{\ell_R(|Q_+|-|Q_-|)} q^{\frac{N(|R_+|+|R_-|+|Q_+|+|Q_-|)}{2}} \\
 &\times q^{\frac{\kappa_{R_+} + \kappa_{R_-}}{2}} \sum_P q^{-N|P|} (-)^{|P|} \hat{C}_{Q_+^T R_+ P}(q) \hat{C}_{Q_-^T R_- P^T}(q). \quad (4.4)
 \end{aligned}$$

The amplitude $\hat{C}_{RPQ}(q)$ is the topological vertex amplitude of [15].⁸ In (4.4) $M(q)$ and $\eta(q)$ are MacMahon and Dedekind functions.

Putting this all together, let us now parameterize the integers ℓ_{R_i} as follows

$$3\ell = \ell_{R_1} + \ell_{R_2} + \ell_{R_3}, \quad 3n = \ell_{R_1} - \ell_{R_3}, \quad 3k = \ell_{R_2} - \ell_{R_3}.$$

It is easy to see that the sum over n and k gives delta functions: at large N

$$Z_{qYM}(\theta_i, g_s) \sim \delta(N(\theta_1 - \theta_3)) \delta(N(\theta_2 - \theta_3)) \times Z_{qYM}^{finite}(\theta, g_s) \quad (4.5)$$

where $\theta_i = \theta$ in the finite piece. As we will show in Sec. 4.2 there is the same δ -function singularity as in the partition function of the bound-states of N D4 branes. There it will be clear that it comes from summing over D2 branes with charges in $H_2(X, \mathbf{D})^\perp$, as mentioned at the beginning of this section. The finite piece in (4.5) is given by

$$Z_{qYM}^{finite}(N, \theta, g_s) = \sum_{m \in \mathbf{Z}} \sum_{P_1, P_2, P_3} (-)^{\sum_{i=1}^3 |P_i|} Z_{P_1, P_2, P_3}^+(t + mg_s) Z_{P_1^T, P_2^T, P_3^T}^+(\bar{t} - mg_s). \quad (4.6)$$

The chiral block in (4.6) is the topological string amplitude on $X = O(-3) \rightarrow \mathbf{P}^2$,

$$\begin{aligned}
 Z_{P_1, P_2, P_3}^+(t) &= \hat{Z}_0(g_s, t) e^{-t_0 \sum_i |P_i|} \\
 &\sum_{R_1, R_2, R_3} e^{-t \sum_i |R_i|} q^{\sum_i \kappa_{R_i}} \hat{C}_{R_2^T R_1 P_1^T}(q) \hat{C}_{R_3^T R_2 P_2^T}(q) \hat{C}_{R_1^T R_3 P_3^T}(q)
 \end{aligned} \quad (4.7)$$

where $t_0 = -\frac{1}{2}Ng_s$ and the Kahler modulus t is (we will return to the meaning of t_0 shortly):

$$t = \frac{3Ng_s}{2} - i\theta.$$

More precisely, the chiral block with trivial ghosts $P_i = 0$,

$$Z_{0,0,0}^+(t, g_s) = Z^{top}(t, g_s)$$

is exactly equal to the perturbative closed topological string partition function for $X = O(-3) \rightarrow \mathbf{P}^2$, as given in [15]. This exactly agrees with the prediction of [1].

The prefactor $\hat{Z}_0(g_s, t)$ is given by

$$\hat{Z}_0(g_s, t) = e^{-\frac{t^3}{18g_s^2}} M^3(q^{-1}) \eta^{\frac{t}{g_s}}(q^{-1}) \theta^{\frac{t}{g_s}}(q)$$

⁸The conventions of this paper and [15] differ, as here $q = e^{-g_s}$, but $q_{there} = e^{g_s}$, consequently the topological vertex amplitude C_{RPQ} of [15] is related to the present one by $\hat{C}_{RPQ}(q) = C_{RPQ}(q^{-1})$.

As explained in [3] the factor $\eta^{\frac{t}{g_s}} \sim \eta^{\frac{3N}{2}}$ comes from bound states of D0 and D4 branes [10] without any D2 brane charge, and moreover, it has only genus zero contribution perturbatively.

$$\eta^{\frac{t}{g_s}} \sim \exp\left(-\frac{\pi^2 t}{6g_s^2}\right) + (\text{non-perturbative})$$

The factor $\theta^{\frac{t}{g_s}}$ comes from the bound states of D4 branes with D2 branes along each of three the non-compact toric legs in the normal direction to the \mathbf{P}^2 , and without any D0 branes. This gives no perturbative contributions

$$\theta^{\frac{t}{g_s}} \sim 1 + (\text{non-perturbative})$$

The subleading chiral blocks correspond to open topological string amplitudes in X with D-branes along the fiber direction to the \mathbf{P}^2 , which can be computed using the topological vertex formalism [15]. The appearance of D-branes was explained in [9] where they were interpreted as non-normalizable modes of the topological string amplitudes on X . The reinterpretation in terms of non-normalizable modes of the topological string theory is a consequence of the open-closed topological string duality on [16]. While this is a duality in the topological string theory, in the physical string theory the open and closed string theory are the same only provided we turn on Ramond-Ramond fluxes. We cannot do this here however, since this would break supersymmetry, and the only correct interpretation is the closed string one.

To make contact with this, define

$$Z^+(U_1, U_2, U_3) = \sum_{R_1, R_2, R_3} Z^+_{R_1, R_2, R_3} \text{Tr}_{R_1} U_1 \text{Tr}_{R_2} U_2 \text{Tr}_{R_3} U_3.$$

where U_i are unitary matrices. This could be viewed as an open topological string amplitude with D-branes, or more physically, as the topological string amplitude, with non-normalizable deformations turned on. These are not most general non-normalizable deformations on X , but only those that preserve torus symmetries – correspondingly they are localized along the non-compact toric legs, just like the topological D-branes that are dual to them are. The non-normalizable modes of the geometry can be identified with [16]

$$\tau_i^n = g_s \text{tr}(U_i^n)$$

where the trace is in the fundamental representation. We can then write (4.6) as

$$Z^{\text{finite}} \sim \int dU_1 dU_2 dU_3 |Z^+(U_1, U_2, U_3)|^2$$

where we integrate over *unitary* matrices *provided* we shift

$$U \rightarrow U e^{-t_0}$$

where $t_0 = -\frac{1}{2} N g_s$. This shift is the attractor mechanism for the non-normalizable modes of the geometry [9]. In terms of the natural variables t_i^n , related by $\tau_i^n = \exp(-t_i^n)$ to τ 's we have

$$t_i^n = n t_0 \tag{4.8}$$

This comes about as follows [9] . First note that size of any 2-cycle C in the geometry should be fixed by the attractor mechanism to equal its intersection with the 4-cycle class $[D]$ of the D4 branes, in this case $[D] = 3N[D_t]$. The relevant 2-cycle in this case is a disk C_0 ending on the topological D-brane. The real part of t_i^n measures the size of an n -fold cover of this disk (there is no chemical potential, i.e. t_0 is real, since there is associated BPS state of finite mass). Then (4.8) follows because

$$\#(C_0 \cap D) = -N.$$

To see this note that in homology, the class $3N[D_t]$ could equally well be represented by $-N$ D-branes on the base \mathbf{P}^2 and the latter has intersection number 1 with C_0 . The factor of n in (4.8) comes about since t^n corresponds to the size of the n -fold cover of the disk.

ii. The general N_i case.

The case $N_1 > N_2 > N_3$ is substantially more involved, and in particular, the large N limit of the amplitudes (3.12)(4.1) is not known. However, as we will explain in the appendix D, *turning off* the $U(1)$ factors of the gauge theory, the large N limit can be computed, and we find a remarkable agreement with the conjecture of [1] .

Let us focus on the leading chiral block of the amplitude. The large N, M limit of the interaction $V_{QR}(M, N)$ (more precisely, the modified version of it to turn off the $U(1)$ charges) is

$$V_{QR} \sim \beta_M q^{\frac{(|Q_+|+|Q_-|)(N-M)}{2}} q^{\frac{(|R_+|+|R_-|)(M-N)}{2}} q^{-\frac{(\kappa_{R_+}+\kappa_{R_-}+\kappa_{Q_+}+\kappa_{Q_-})}{2}} W_{Q_+R_+}(q) W_{Q_-R_-}(q) \tag{4.9}$$

where

$$\beta_M = q^{(\rho_M^2 + \frac{M}{24})} M(q^{-1}) \eta^M(q^{-1}) \theta^M(q).$$

In (4.9) the W_{PR} is related to the topological vertex amplitude as $W_{PR}(q) = (-)^{|P|+|R|} \times \hat{C}_{0P^T R}(q) q^{\kappa_R/2}$. It is easy to see that for $N = M$ this agrees with the large N limit of the simpler form of the V_{RQ} amplitude in (3.13). It is easy to see that that the leading chiral block of (4.1) is

$$Z_{qYM} \sim Z_{0,0,0}^+(t) Z_{0,0,0}^-(\bar{t}) \tag{4.10}$$

where $Z_{0,0,0}^+(t)$ is

$$Z_{0,0,0}^+ = \hat{Z}_0 \sum_{R_+, Q_+, P_+} W_{R_+Q_+}(q) W_{Q_+P_+}(q) W_{P_+R_+}(q) e^{-t(|R_+|+|Q_+|+|P_+|)}$$

which is the closed topological string amplitude on X . In particular, this agrees with the amplitude in (4.7) . In the present context, the Kahler modulus t is given by

$$t = \frac{1}{2}(N_1 + N_2 + N_3)g_s - i\theta.$$

This is exactly as dictated by the attractor mechanism corresponding to the divisor $[D] = (N_1 + N_2 + N_3)[D_t]!$

The higher chiral blocks will naturally be more involved in this case. Some of the intersection numbers fixing the attractor positions of ghost branes are ambiguous, and correspondingly, far more complicated configurations of non-normalizable modes are expected.

Recently, phase transitions in 2d qYM theory on \mathbf{P}^1 were studied in [17][18][19][20][21], where it was shown that the theory has a phase transition when the Chern class of the normal bundle to the branes is not positive. It would be very interesting to study the phase transitions in the present case.

4.2 Branes on local \mathbf{P}^2

In this and subsequent section we will discuss the degeneracies of BPS states that follow from (4.1). Using the results of (3.17) and (3.18) or by direct computation, it is easy to see that Z_{qYM} is a modular form. Its form however is the simplest in the case

$$N_1 = N_2 = N_3 = N,$$

so let us treat this first.

i. Degeneracies for $N_i = N$.

In this case, the form of the partition function written in (4.2) is more convenient. By trading the sum over representations and over the Weyl-group, as in (3.18), for sums over the weight lattices, the partition function of BPS states is

$$Z_{qYM}(N, \theta_i, g_s) = \beta \sum_{w \in S_N} (-)^w \sum_{n_1, n_2, n_3 \in \mathbf{Z}^N} q^{\frac{1}{2} \sum_{i=1}^3 n_i^2} q^{w(n_1) \cdot n_2 + n_2 \cdot n_3 + n_3 \cdot n_1} e^{i \sum_{i=1}^3 \theta_i e(N) \cdot n_i} \quad (4.11)$$

where $e(N) = (1, \dots, 1)$ and $\beta = \alpha \theta^{3N}(q)$. The amplitudes depend on the permutations w only through their conjugacy classes, consequently we have:

$$Z_{qYM} = \beta \sum_{\vec{K}} d(\vec{K}) Z_{K_1} \times \dots \times Z_{K_r} \quad (4.12)$$

where \vec{K} labels a partition of N into natural numbers $N = \sum_{a=1}^r K_a$, and $d(\vec{K})$ is the number of elements in the conjugacy class of S_N , the permutation group of N elements, corresponding to having r cycles of length K_a , $a = 1, \dots, r$, and

$$Z_K(\theta_i, g_s) = (-)^{w_K} \sum_{n_1, n_2, n_3 \in \mathbf{Z}^K} q^{\frac{1}{2} \sum_{i=1}^3 n_i^2} q^{w_K(n_1) \cdot n_2 + n_2 \cdot n_3 + n_3 \cdot n_1} e^{i \sum_{i=1}^3 \theta_i e(K) \cdot n_i} \quad (4.13)$$

Here w_K stands for cyclic permutation of K elements. Note that the form of the partition function (4.12) suggests that Z_{qYM} is counting not only BPS bound states, but also contains contribution from marginally bound states corresponding to splitting of the $U(N)$ to

$$U(N) \rightarrow U(K_1) \times U(K_2) \times \dots \times U(K_r)$$

In each of the sectors, the quadratic form is degenerate. The contribution of bound states of N branes Z_N diverges as

$$Z_N(\theta_i, g_s) \sim \sum_{m_1, m_2 \in \mathbf{Z}} e^{iNm_1(\theta_1 - \theta_3)} e^{iNm_2(\theta_2 - \theta_3)} = \delta(N(\theta_1 - \theta_3)) \delta(N(\theta_2 - \theta_3))$$

This is exactly the type of the divergence we found at large N in the previous subsection. This divergence should be related to summing over D_2 branes with charges in $H_2(D, \mathbf{Z})^\perp$ – these apparently completely decouple from the rest of the theory.

More precisely, writing $U(N) = U(1) \times SU(N)/\mathbf{Z}_N$, this will have a sum over 't Hooft fluxes which are correlated with the fluxes of the $U(1)$. Then, Z_N is a sum over sectors of different N -ality,

$$Z_N(\theta_i, g_s) = (-)^{w_N} \sum_{L_i=0}^{N-1} \sum_{\ell_i \in \mathbf{Z} + \frac{L_i}{N}} q^{\frac{N}{2}(\ell_1 + \ell_2 + \ell_3)^2} e^{iN \sum_i \theta_i \ell_i} \sum_{m \in \mathbf{Z}^{3(N-1)} + \vec{\xi}(L_i)} q^{\frac{1}{2} m^T \mathcal{M}_N m}$$

where \mathcal{M}_N is a non-degenerate $3(N-1) \times 3(N-1)$ matrix with integer entries and $\vec{\xi}_i$ is a shift of the weight lattice corresponding to turning on 't Hooft flux. Explicitly,

$$\xi_i^a = \frac{N-a}{N} L_i, \quad i = 1, 2, 3 \quad a = 0, \dots, N-1$$

where \mathcal{M}_N is $3(N-1) \times 3(N-1)$ matrix

$$\mathcal{M}_N = \begin{pmatrix} M_N & W_N & M_N \\ W_N^T & M_N & M_N \\ M_N & M_N & M_N \end{pmatrix} \quad (4.14)$$

whose entries are

$$M_N = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \quad (4.15)$$

and

$$W_N = \begin{pmatrix} -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & \cdot & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ -1 & 0 & 0 & \dots & 0 & -1 \end{pmatrix} \quad (4.16)$$

We can express Z_N in terms of Θ -functions

$$Z_N(\theta_i, g_s) = (-)^{w_N} \delta(N(\theta_1 - \theta_3)) \delta(N(\theta_2 - \theta_3)) \sum_{L_i=0}^{N-1} \Theta_1[a(L_i), b](\tau) \Theta_{3N-3}[\mathbf{a}(L_i), \mathbf{b}](\hat{\tau})$$

where

$$\Theta_k[a, b](\tau) = \sum_{n \in \mathbf{Z}^k} e^{\pi i \tau (n+a)^2} e^{2\pi i n b}$$

and

$$\begin{aligned} \tau &= \frac{i g_s}{2\pi} N, & \hat{\tau} &= \frac{i g_s}{2\pi} \mathcal{M}_N \\ a &= \frac{L_1 + L_2 + L_3}{N}, & b &= \frac{N}{2\pi} \theta, & \mathbf{a}_L &= \vec{\xi}(L), & \mathbf{b} &= \mathbf{0}, \end{aligned}$$

The origin of the divergent factor we found is now clear: from the gauge theory perspective it simply corresponds to a partition function of a $U(1) \in U(N)$ gauge theory on a 4-manifold whose intersection matrix is degenerate: $\#(C_i \cap C_j) = 1$, $i, j = 1, 2, 3$. More precisely, to define the intersection form of the reducible four-cycle D , note that D is homologous to the (punctured) \mathbf{P}^2 in the base, with precisely the intersection form at hand. The contribution of marginally bound states with multiple $U(1)$ factors have at first sight a worse divergences, however these can be regularized by ζ -function regularization to zero.⁹ This is a physical choice, since in these sectors we expect the partition function to vanish due to extra fermion zero modes [10][22].

To extract the black hole degeneracies we use that the matrix \mathcal{M}_N is non-degenerate and do modular S-transformation using

$$\Theta[a, b](\tau) = \det(\tau)^{-\frac{1}{2}} e^{2\pi i a b} \Theta[b, -a](-\tau^{-1})$$

This brings Z_N to the form

$$\begin{aligned} Z_N(\theta_i, g_s) &= \delta(N(\theta_1 - \theta_3)) \delta(N(\theta_2 - \theta_3)) (-)^{w_N} \left(\frac{2\pi}{N g_s} \right)^{\frac{1}{2}} \left(\frac{2\pi}{g_s} \right)^{\frac{3(N-1)}{2}} \det^{-\frac{1}{2}} \mathcal{M}_N \\ &\sum_{L_i=0}^{N-1} \sum_{\ell \in \mathbf{Z}} e^{-\frac{2\pi^2}{N g_s} (\ell + \frac{N\theta}{2\pi})^2} e^{-\frac{2\pi i (L_1 + L_2 + L_3)}{N} \ell} \sum_{m \in \mathbf{Z}^{3(N-1)}} e^{-\frac{2\pi^2}{g_s} m^T \mathcal{M}_N^{-1} m} e^{-2\pi i m \cdot \xi(L_i)} \end{aligned}$$

where \mathcal{M}_N is the matrix in (4.14) .

ii. Degeneracies for $N_1 > N_2 > N_3$.

When the number of branes is not equal the partition sum Z_{qYM} is substantially more complicated. By manipulations similar to the ones in appendix B, Z_{qYM} can be written as:

$$\begin{aligned} Z_{qYM} &= \alpha \theta^{N_2 + 2N_3} (q) \sum_{\nu \in S_{N_1}} (-)^\nu \sum_{n_1 \in \mathbf{Z}^{N_1}} \sum_{n_2 \in \mathbf{Z}^{N_2}} \sum_{n_3 \in \mathbf{Z}^{N_3}} q^{\frac{1}{2}(n_1^2 - n_3^2)} q^{n_2 \cdot \nu(n_1) + n_3 \cdot n_2 + n_3 \cdot n_1} \\ &q^{-\frac{1}{2} n_1 (\nu^{-1} \hat{P}_{N_1|N_2} \nu) n_1} q^{-\frac{1}{2} n_2 \hat{P}_{N_2|N_3} n_2} q^{-\frac{1}{2} n_1 \hat{P}_{N_1|N_3} n_1} \end{aligned}$$

⁹For example, $Z_{N-M}(\theta_i, g_s) Z_M(\theta_i, g_s) \sim \delta(k(\theta_1 - \theta_3)) \times \sum_{n \in \mathbf{Z}} 1 \times \delta(k(\theta_2 - \theta_3)) \times \sum_{n \in \mathbf{Z}} 1$. where k is the least common divisor of N, M . Using $\zeta(2s) = \sum_{n=1}^{\infty} 1/n^{2s}$, where $\zeta(0) = -\frac{1}{2}$, we can regularize $\sum_{n \in \mathbf{Z}} 1 = 0$.

$$q^{-\nu(n_1) \cdot (\rho_{N_1} - \rho_{N_2})} q^{-n_2 \cdot (\rho_{N_2} - \rho_{N_3})} q^{-n_1 \cdot (\rho_{N_1} - \rho_{N_3})} e^{i\theta_1 e(N_1) \cdot n_1 + i\theta_2 e(N_2) \cdot n_2 + i\theta_3 e(N_3) \cdot n_3}$$

where operator $\hat{P}_{N|M}$ projects N -dimensional vector on its first M components.

For example, consider $N_1 = 3, N_2 = 2, N_3 = 1$. In this case there are six terms in the sum

$$\begin{aligned} \nu_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \nu_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \nu_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \nu_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \nu_5 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & \nu_6 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

In this simple case Z_{qYM} has the form

$$Z_{qYM} = \alpha \theta^4 (q') \left(\frac{\pi}{g_s} \right)^2 (Z_1 - Z_2 - Z_3 - Z_4 + Z_5 + Z_6) \quad q' = e^{-\frac{\pi^2}{g_s}}$$

where

$$Z_i = \left(\frac{2\pi}{g_s} \right)^3 \det^{-\frac{1}{2}} \mathcal{M}_{(i)} \sum_{f \in \mathbf{Z}^6} e^{-\frac{2\pi^2}{g_s} (f + \Lambda_{(i)})^T \mathcal{M}_{(i)}^{-1} (f + \Lambda_{(i)})}$$

where non-degenerate matrices $\mathcal{M}_{(i)}$ for $i = 1, \dots, 6$ are given by

$$\begin{aligned} \mathcal{M}_{(1)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \end{pmatrix}, & \mathcal{M}_{(2)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix} \\ \mathcal{M}_{(3)} &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}, & \mathcal{M}_{(4)} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix} \\ \mathcal{M}_{(5)} &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}, & \mathcal{M}_{(6)} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix} \end{aligned}$$

and vectors $\Lambda_{(i)}$ for $i = 1, \dots, 6$ have components

$$\Lambda_{(1)} = \frac{1}{2\pi} (\theta_1, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3) + \frac{ig_s}{2\pi} (2, -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 0)$$

$$\begin{aligned}
 \Lambda_{(2)}^a &= \Lambda_{(1)}^a, \quad a = 1, \dots, 6 \\
 \Lambda_{(3)}^1 &= \frac{1}{2\pi}(\theta_1, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3) + \frac{ig_s}{2\pi}\left(\frac{1}{2}, -\frac{3}{2}, 1, \frac{1}{2}, -\frac{1}{2}, 0\right) \\
 \Lambda_{(6)}^a &= \Lambda_{(3)}^a, \quad a = 1, \dots, 6 \\
 \Lambda_{(4)}^1 &= \frac{1}{2\pi}(\theta_1, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3) + \frac{ig_s}{2\pi}\left(\frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0\right) \\
 \Lambda_{(5)}^a &= \Lambda_{(4)}^a, \quad a = 1, \dots, 6
 \end{aligned}$$

5. Branes and black holes on local $\mathbf{P}^1 \times \mathbf{P}^1$

For our second example, we will take a noncompact Calabi-Yau threefold X which is a total space of canonical line bundle K over the base $\mathcal{B} = \mathbf{P}_B^1 \times \mathbf{P}_F^1$

$$X = K \rightarrow \mathbf{P}_B^1 \times \mathbf{P}_F^1$$

where $K = \mathcal{O}(-2, -2)$. The linear sigma model whose Higgs branch is X has chiral fields X_i , $i = 0, \dots, 4$ and two $U(1)$ gauge fields $U(1)_B$ and $U(1)_F$ under which the chiral fields have charges $(-2, 1, 0, 1, 0)$ and $(-2, 0, 1, 0, 1)$. The corresponding D-term potentials are

$$\begin{aligned}
 |X_1|^2 + |X_3|^2 &= 2|X_0|^2 + r_B \\
 |X_2|^2 + |X_4|^2 &= 2|X_0|^2 + r_F
 \end{aligned}$$

The $H^2(X, Z)$ is generated by two classes $[D_F]$ and $[D_B]$. Correspondingly, there are two complexified Kahler moduli t_B and t_F , $t_B = r_B - i\theta_B$ and $t_F = r_F - i\theta_F$. There are 4 ample divisors invariant under the T^3 torus actions corresponding to setting

$$D_i : X_i = 0, \quad i = 1, 2, 3, 4$$

We have that $[D_1] = [D_3] = [D_B]$ and $[D_2] = [D_4] = [D_F]$. We take N_1 and N_2 D4 branes on D_1 and D_3 , and M_1 and M_2 D4 branes on D_2 and D_4 respectively, corresponding to a divisor

$$D = N_1 D_1 + M_1 D_2 + N_2 D_3 + M_2 D_4$$

Since the topology of each D_i is $\mathcal{O}(-2) \rightarrow \mathbf{P}^1$ we will get four copies of qYM theory of \mathbf{P}^1 with ranks $N_{1,2}$ and $M_{1,2}$. In addition, from the matter at intersection we get 4 sets of insertions of observables (3.3). All together, and assuming $N_{1,2} \geq M_{1,2}$, we have

$$\begin{aligned}
 Z_{qYM} &= \gamma \sum_{\mathcal{R}_1, \mathcal{R}_2, \mathcal{Q}_1, \mathcal{Q}_2} V_{\mathcal{Q}_1 \mathcal{R}_1} V_{\mathcal{Q}_2 \mathcal{R}_2} V_{\mathcal{R}_1 \mathcal{Q}_2} V_{\mathcal{R}_2 \mathcal{Q}_1} q^{\sum_{i=1}^2 C_2(\mathcal{R}_i) + C_2(\mathcal{Q}_i)} \\
 &\quad e^{i\theta_{B,1} C_1(\mathcal{R}_1) + i\theta_{B,2} C_1(\mathcal{R}_2)} e^{i\theta_{F,1} C_1(\mathcal{Q}_1) + i\theta_{F,2} C_1(\mathcal{Q}_2)}.
 \end{aligned} \tag{5.1}$$

Above, $\mathcal{R}_1, \mathcal{R}_2$ are representations of $U(N_1)$ and $U(N_2)$ and $\mathcal{Q}_1, \mathcal{Q}_2$ are representations of $U(M_1)$ and $U(M_2)$, respectively.

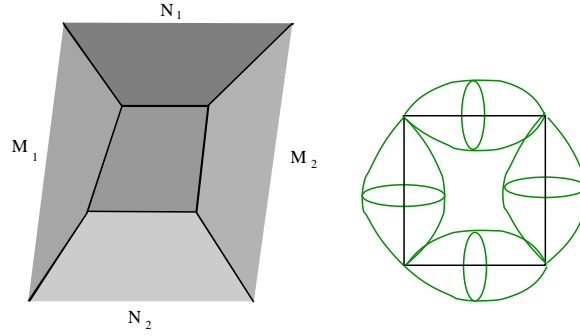


Figure 4: The base of the local $\mathbf{P}^1 \times \mathbf{P}^1$. The numbers of D4 branes wrapping the torus invariant non-compact 4-cycles are specified. This corresponds to qYM theory on the necklace of 4 \mathbf{P}^1 's with ranks M_1, N_1, M_2 , and N_2 .

In principle, because $\dim(H^2(D, \mathbf{Z})) = 4$, there are 4 different chemical potentials that we can turn on for the D2 branes, corresponding to $\theta_{B,i}, \theta_{F,i}$. In X however, there are only two independent classes, $\dim(H^2(D, \mathbf{Z})) = 2$, in particular

$$[\mathbf{P}_{B,1}^1] - [\mathbf{P}_{B,2}^1] = 0, \quad [\mathbf{P}_{F,1}^1] - [\mathbf{P}_{F,2}^1] = 0$$

We should turn off the chemical potentials for those states that can decay when the YM theory is embedded in string theory, by putting

$$\theta_{B,1} = \theta_{B,2}, \quad \theta_{F,1} = \theta_{F,2}. \quad (5.2)$$

For the most part, we will keep the chemical potentials arbitrary, imposing (5.2) at the end. The prefactor γ is

$$\begin{aligned} \gamma &= q^{-(2\rho_{M_1}^2 + \frac{M_1}{12})} \\ &\times q^{-(2\rho_{M_2}^2 + \frac{M_2}{12})} q^{-\frac{1}{96} \left((N_1+N_2)^3 + (M_1+M_2)^3 - 3(N_1+N_2)^2(M_1+M_2) - 3(M_1+M_2)^2(N_1+N_2) \right)} \\ &\times e^{\frac{\theta_B \theta_F (N_1+N_2+M_1+M_2)}{4gs}} \end{aligned}$$

In the next subsections we will first take the large N limit of the qYM partition function, and then consider the modular properties of the exact amplitude to compute the degeneracies of the BPS bound states.

5.1 Black holes on local $\mathbf{P}^1 \times \mathbf{P}^1$.

We will now take the large N limit of Z_{qYM} in (5.1) and show that this is related to the topological string on X in accordance with the [1] conjecture.

i. The $N_1 = N_2 = N = M_1 = M_2$ case.

In this case, we can use the simpler form of the vertex amplitude in (3.12) to write the q-deformed Yang-Mills partition function as:

$$Z_{qYM} = \gamma' \sum_{\mathcal{R}_{1,2}, \mathcal{Q}_{1,2} \in U(N)}$$

$$\begin{aligned}
 & S_{\mathcal{R}_1 \bar{\mathcal{Q}}_1}(g_s, N) S_{\mathcal{Q}_1 \bar{\mathcal{R}}_2}(g_s, N) S_{\mathcal{R}_2 \bar{\mathcal{Q}}_2}(g_s, N) S_{\mathcal{Q}_2 \bar{\mathcal{R}}_1}(g_s, N) \\
 & \times e^{i \sum_i \theta_{B,i} C_1(\mathcal{R}_i) + i \theta_{F,i} C_1(\mathcal{Q}_i)}.
 \end{aligned} \tag{5.3}$$

where $\gamma' = \gamma \theta^{4N}(q)$. Using the large N expansion for S-matrix (4.4) and parametrizing the $U(1)$ charges ℓ_{R_i} of the representations \mathcal{R}_i as follows

$$2\ell_B = \ell_{R_1} + \ell_{R_2}, \quad 2\ell_F = \ell_{Q_1} + \ell_{Q_2}, \quad 2n_B = \ell_{R_1} - \ell_{R_2}, \quad 2n_F = \ell_{Q_1} - \ell_{Q_2}, \tag{5.4}$$

we find that the sum over $n_{B,F}$ gives delta functions

$$Z_{qYM}(N, g_s, \theta_{B,i}, \theta_{F,i}) \sim \delta(N(\theta_{B,1} - \theta_{B,2})) \delta(N(\theta_{F,1} - \theta_{F,2})) Z_{qYM}^{finite}(N, g_s, \theta_B, \theta_F)$$

where

$$\begin{aligned}
 Z_{qYM}^{finite} & \sim \sum_{m_B, m_F \in \mathbf{Z}} \sum_{P_1, \dots, P_4} (-)^{\sum_{i=1}^4 |P_i|} \\
 & Z_{P_1, \dots, P_4}^+(t_B + m_B g_s, t_F + m_F g_s) \\
 & Z_{P_1^T, \dots, P_4^T}^+(\bar{t}_B - m_B g_s, \bar{t}_F - m_F g_s)
 \end{aligned} \tag{5.5}$$

In (5.5) the chiral block $Z_{P_1, \dots, P_4}^+(t_B, t_F)$ is given by

$$\begin{aligned}
 Z_{P_1, \dots, P_4}^+(t_B, t_F) & = \hat{Z}_0(g_s, t_B, t_F) e^{-t_0 \sum_{i=1}^4 |P_i|} \sum_{R_1, R_2, Q_1, Q_2} e^{-t_B(|R_1| + |R_2|)} e^{-t_F(|Q_1| + |Q_2|)} \\
 & \times q^{\frac{1}{2} \sum_{i=1,2} \kappa_{R_i} + \kappa_{Q_i}} \hat{C}_{Q_1^T R_1 P_1}(q) \hat{C}_{R_2^T Q_1 P_2}(q) \hat{C}_{Q_2^T R_2 P_3}(q) \hat{C}_{R_1^T Q_2 P_4}(q)
 \end{aligned} \tag{5.6}$$

where Kahler moduli are

$$t_B = g_s N - i\theta_B, \quad t_F = g_s N - i\theta_F.$$

The leading chiral block $Z_{0, \dots, 0}^+$ is the closed topological string amplitude on X . The Kahler moduli of the base \mathbf{P}_B^1 and the fiber \mathbf{P}_F^1 are exactly the right values fixed by the attractor mechanism: since the divisor D that the D4 brane wraps is in the class $[D] = 2N[D_F] + 2N[D_B]$. As we discussed in the previous section in detail, the other chiral blocks (5.6) correspond to having torus invariant non-normalizable modes excited along the four non-compact toric legs in the normal directions to the base \mathcal{B} . Moreover the associated Kahler parameters should also be fixed by the attractor mechanism – as discussed in the previous section, we can think of these as the open string moduli corresponding to the ghost branes. The open string moduli are complexified sizes of holomorphic disks ending on the ghost branes and these can be computed using the Kahler form on X . Since the net D4 brane charge is the same as that of $-N$ branes wrapping the base, and the intersection number of the disks C_0 ending on the topological D-branes with the base is $\#(C_0 \cap \mathcal{B}) = 1$, so the size of all the disks ending on the branes should be $t_0 = -\frac{1}{2} N g_s$, which is in accord with (5.6). The prefactor in (5.6) is

$$\hat{Z}_0(g_s, t_B, t_F) = e^{\frac{1}{24g_s^2}(t_F^3 + t_B^3 - 3t_F^2 t_B - 3t_B^2 t_F)} M^4(q^{-1}) \eta^{\frac{t_B + t_F}{g_s}}(q^{-1}) \theta^{\frac{t_B + t_F}{g_s}}(q)$$

As discussed before, the eta and theta function pieces contribute only to the genus zero amplitude, and to the non-perturbative terms.

ii. The general $N_{1,2}, M_{1,2}$ case.

We will assume here $N_i > M_j$, $i, j = 1, 2$. Using the large N, M limit of $V_{\mathcal{RQ}}(N, M)$ with $U(1)$ charges turned off (see Appendix D) we find that the leading chiral block of the YM partition function is

$$Z_{qYM} \sim Z_{0,\dots,0}^+(t_B, t_F) Z_{0,\dots,0}^-(\bar{t}_B, \bar{t}_F)$$

where $Z_{0,\dots,0}^+(t_B, t_F)$ is precisely the topological closed string partition function on local $\mathbf{P}^1 \times \mathbf{P}^1$ [15] :

$$Z_{0,\dots,0}^+ = \hat{Z}_0 \sum_{Q_1^+, Q_2^+, R_1^+, R_2^+} W_{Q_1^+ R_1^+}(q) W_{Q_1^+ R_2^+}(q) W_{Q_2^+ R_1^+}(q) W_{Q_2^+ R_2^+}(q) \\ \times e^{-t_F(|Q_1^+|+|Q_2^+|)} e^{-t_B(|R_1^+|+|R_2^+|)}$$

It is easy to see that this agrees with the amplitude given in (5.6) . Moreover, the Kahler parameters are exactly as predicted by the attractor mechanism corresponding to having branes on a divisor class

$$[D] = (N_1 + N_2)[D_B] + (M_1 + M_2)[D_F].$$

Namely,

$$t_B = \frac{1}{2}(M_1 + M_2)g_s - i\theta_B, \quad t_F = \frac{1}{2}(N_1 + N_2)g_s - i\theta_F.$$

Note that the normal bundle to each of the divisor D_i is trivial, so the size of the corresponding \mathbf{P}^1 in $D_i = O(-2) \rightarrow \mathbf{P}^1$ is independent of the number of branes on D_i , but it does depend on the number of branes on the adjacent faces which have intersection number 1 with the \mathbf{P}^1 .

It would be interesting to study the structure of the higher chiral blocks. In this case we expect the story to be more complicated, in particular because some of the intersection numbers that compute the attractor values of the brane moduli are now ambiguous.

5.2 Branes on local $\mathbf{P}^1 \times \mathbf{P}^1$

We will content ourselves with considering $N_{1,2} = M_{1,2} = N$ case, the more general case working in similar ways to the local \mathbf{P}^2 case. The partition function (5.3) may be written as

$$Z_{qYM}(N, \theta_i, g_s) = \gamma' \sum_{w \in S_N} (-)^w \sum_{n_1, \dots, n_4 \in \mathbb{Z}^N} q^{w(n_1) \cdot n_2 + n_2 \cdot n_3 + n_3 \cdot n_4 + n_4 \cdot n_1} e^{i \sum_{i=1}^4 \theta_i e(N) \cdot n_i} \quad (5.7)$$

where $e(N) = (1, \dots, 1)$. As before in the case of local \mathbf{P}^2 , the bound states of N D4-branes are effectively counted by the Z_N term, i.e. the term with $w = w_N$. Like in that case, Z_N is again a sum over sectors of different N -ality,

$$Z_N(\theta_i, g_s) = \gamma'(-)^{w_N} \sum_{L_1, \dots, L_4=0}^{N-1} \sum_{\ell_i \in \mathbf{Z} + \frac{L_i}{N}} q^{N(\ell_1+\ell_3)(\ell_2+\ell_4)} e^{iN \sum_{i=1}^4 \theta_i \ell_i} \sum_{m \in \mathbf{Z}^{4(N-1)} + \vec{\xi}(L_i)} q^{\frac{1}{2} m^T \mathcal{M} m}$$

where \mathcal{M} is a non-degenerate $4(N-1) \times 4(N-1)$ matrix with integer entries and $\vec{\xi}_i$ is a shift of the weight lattice corresponding to turning on 't Hooft flux.

More explicitly,

$$\xi_i^a = \frac{N-a}{N} L_i, \quad i = 1, \dots, 4 \quad a = 0, \dots, N-1$$

\mathcal{M} is $4(N-1) \times 4(N-1)$ matrix

$$\mathcal{M} = \begin{pmatrix} 0 & W_N & 0 & M_N \\ W_N^T & 0 & M_N & 0 \\ 0 & M_N & 0 & M_N \\ M_N & 0 & M_N & 0 \end{pmatrix} \tag{5.8}$$

whose entries are $(N-1) \times (N-1)$ matrices

$$M_N = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \tag{5.9}$$

and

$$W_N = \begin{pmatrix} -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix} \tag{5.10}$$

We can express Z_N in terms of Θ -functions

$$Z_N(\theta_i, g_s) = \gamma'(-)^{w_N} \delta(N(\theta_{B,1} - \theta_{B,2})) \delta(N(\theta_{F,1} - \theta_{F,2})) \sum_{L_1, \dots, L_4=0}^{N-1} \Theta_2[a(L_i), b](\tau) \Theta_{4N-4}[\mathbf{a}(L_i), \mathbf{b}](\hat{\tau})$$

where

$$\Theta_k[a, b](\tau) = \sum_{n \in \mathbf{Z}^k} e^{\pi i \tau (n+a)^2} e^{2\pi i n b}$$

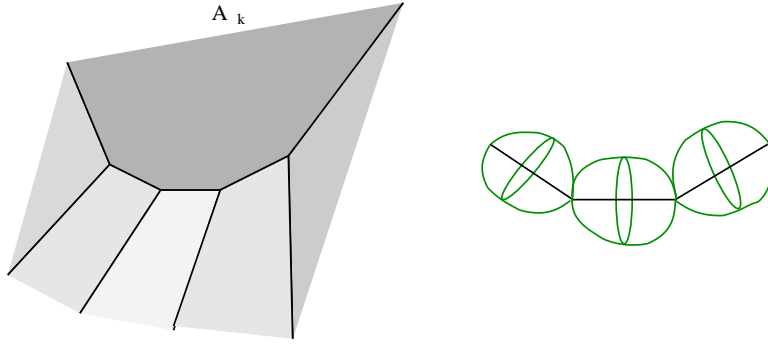


Figure 5: N D4-branes are wrapped on A_k type ALE space in $A_k \times \mathbf{C}$, for $k = 3$. The D-brane partition function is computed by $U(N)$ qYM theory on a chain of 3 \mathbf{P}^1 's.

and

$$\tau = \frac{ig_s}{2\pi} N \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\tau} = \frac{ig_s}{2\pi} \mathcal{M}$$

and

$$a = \left(\frac{L_1 + L_3}{N}, \frac{L_2 + L_4}{N} \right), \quad b = \left(\frac{N}{2\pi} \theta_B, \frac{N}{2\pi} \theta_F \right) \quad \mathbf{a}_L = \vec{\xi}(L), \quad \mathbf{b} = \mathbf{0},$$

To extract black hole degeneracies we use that matrix \mathcal{M} is non-degenerate and do modular S-transformation using

$$\Theta[a, b](\tau) = \det(\tau)^{-\frac{1}{2}} e^{2\pi i ab} \Theta[b, -a](-\tau^{-1})$$

After modular S-transformation Z_N is brought to the form

$$Z_N(\theta, g_s) = \gamma' \delta(N(\theta_{B,1} - \theta_{B,2})) \delta(N(\theta_{F,1} - \theta_{F,2})) (-)^{w_N} \left(\frac{2\pi}{Ng_s} \right) \left(\frac{2\pi}{g_s} \right)^{\frac{4(N-1)}{2}} \det^{-\frac{1}{2}} \mathcal{M} \\ \sum_{L_1, \dots, L_4=0}^{N-1} \sum_{\ell, \ell' \in \mathbf{Z}} e^{-\frac{\pi^2}{Ng_s} (\ell + \frac{N\theta_B}{2\pi}) (\ell' + \frac{N\theta_F}{2\pi})} e^{-\frac{2\pi i(L_1 + L_3)}{N} \ell} e^{-\frac{2\pi i(L_2 + L_4)}{N} \ell'} \\ \sum_{m \in \mathbf{Z}^{4(N-1)}} e^{-\frac{2\pi^2}{g_s} m^T \mathcal{M}^{-1} m} e^{-2\pi i m \cdot \xi(L_i)}$$

6. Branes and black holes on A_k ALE space

Consider the local toric Calabi Yau X which is A_k ALE space times \mathbf{C} . This can be thought of as the limit of the usual ALE fibration over \mathbf{P}^1 as the size of the base \mathbf{P}^1 goes to ∞ . In this section we will consider black holes obtained by wrapping N D4 branes on the ALE space.

This example will have a somewhat different flavor than the previous two, so we will discuss the D4 brane gauge theory on a bit more detail. On the one hand, the theory on the D4 brane is a topological $U(N)$ Yang-Mills theory on A_k ALE space which has been studied previously [23, 10]. On the other hand, the A_k ALE space has T^2 torus symmetries, so we should be able to obtain the corresponding partition function by an appropriate computation in the two dimensional qYM theory. We will start with the second perspective, and make contact with [23][10] later.

As in [3] and in section 3, our strategy will be to cut the four manifold into pieces where the theory is simple to solve, and then glue the pieces back together. The A_k type ALE space can be obtained by gluing together $k + 1$ copies of \mathbf{C}^2 . Correspondingly, we should be able to obtain YM amplitudes on the ALE space by sewing together amplitudes on \mathbf{C}^2 . Moreover, since the \mathbf{C}^2 and the ALE space have T^2 isometries, the 4d gauge theory computations should localize to fixed points of these isometries, and these are bundles with second Chern class localized at the vertices, and first Chern class along the edges.

Viewed as a manifold fibered by 2-tori T^2 , \mathbf{C}^2 contains two disks, say C_{base} and C_{fiber} that are fixed by torus action (see figure 2 by way of example). Viewed as a line bundle over a disk C_{base} as a base, the $U(1)$ isometry of the fiber allows us to do some gauge theory computations in the qYM theory on C_{base} . In particular, if the bundle is flat the qYM partition function on a disk (3.7) with holonomy $U = \exp(i \oint A)$ fixed on the boundary of the C_{base} fixed and no insertions is¹⁰

$$Z(C)(U) = \sum_{\mathcal{R}} e^{i\theta C_1(\mathcal{R})} S_{0\mathcal{R}}(N, g_s) Tr_{\mathcal{R}} U.$$

What is the four dimensional interpretation of this? The sum over \mathcal{R} in the above corresponds to summing over the four dimensional $U(N)$ gauge fields with

$$\int_{fiber} F_a = \mathcal{R}_a g_s, \quad a = 1, \dots, N, \quad (6.1)$$

where \mathcal{R}_a are the lengths of the rows in the Young tableau of \mathcal{R} .¹¹ This is because on the one hand

$$S_{0\mathcal{R}}(N, g_s) = \langle Tr_{\mathcal{R}} e^{i \oint A} \rangle. \quad (6.2)$$

and on the other $\oint A_a = \int_{base} F_a$ is conjugate to $\oint A_a = \int_{fiber} F_a$, so inserting (6.2) shifts F as in (6.1). The unusual normalization of F has to do with the fact that qYM directly computes the magnetic, rather than the electric partition function: In gluing two disks to get an \mathbf{P}^1 we sum over all \mathcal{R} 's labelling the bundles of the S-dual theory over the \mathbf{P}^1 .

If we are to use 2d qYM theory to compute the $\mathcal{N} = 4$ partition function on ALE space, we must understand what in the 2d language is computing the partition function on \mathbf{C}^2 with

$$\int_{fiber} F_a = \mathcal{R}_a g_s, \quad \int_{base} F_a = \mathcal{Q}_a g_s, \quad a = 1, \dots, N, \quad (6.3)$$

since clearly, what we call the “base” here versus the “fiber” is a matter of convention. Using once more the fact that Φ and $\oint A$ are conjugate, turning on $\int_{base} F_a = \mathcal{Q}_a g_s$ corresponds to inserting $Tr_{\mathcal{Q}} e^{-i\Phi}$ at the point on C_{base} where it intersects C_{fiber} . Thus, turning on (6.3) corresponds to computing $\langle Tr_{\mathcal{Q}} e^{-i\Phi} Tr_{\mathcal{R}} e^{i \oint A} \rangle$. This is an amplitude we already know:

$$S_{\mathcal{Q}\mathcal{R}}(N, g_s) = \langle Tr_{\mathcal{Q}} e^{-i\Phi} Tr_{\mathcal{R}} e^{i \oint A} \rangle. \quad (6.4)$$

¹⁰More precisely, as we explained in section 3, the coordinate U is ambiguous since the choice of cycle which remains finite is ambiguous. This ambiguity relates to the choice of the normal bundle to the disk, and the present choice corresponds to picking this bundle to be trivial, which is implicit in the amplitude.

¹¹To be more precise, \mathcal{R}_a in (6.1) is shifted by $\frac{1}{2}(N + 1) - a$.

Alternatively, the amplitude on \mathbf{C}^2 with arbitrary boundary conditions (6.3) on the base and on the fiber is

$$\sum_{\mathcal{R}, \mathcal{Q}} S_{\mathcal{R}\mathcal{Q}}(N, g_s) \text{Tr}_{\mathcal{R}} U \text{Tr}_{\mathcal{Q}} V \quad (6.5)$$

We then glue the pieces together using the usual local rules. The only thing we have to remember is that the normal bundle to each \mathbf{P}^1 is $O(-2)$, and that at the “ends” we should turn the fields off. In computing (6.4) we used the coordinates in which \mathbf{C}^2 is a trivial fibration over both C_{fiber} and C_{base} , and therefore to get the first Chern class of the normal bundle to come out to be -2 , we must along each of them insert annuli with $O(-2)$ bundle over them. This gives:

$$Z = \sum_{\mathcal{R}_1 \dots \mathcal{R}_k} S_{0\mathcal{R}_{(1)}} S_{\mathcal{R}_{(1)}\mathcal{R}_{(2)}} \dots S_{\mathcal{R}_{(k)}0} q^{\sum C_2(\mathcal{R}_{(j)})} e^{i \sum \theta_j |\mathcal{R}_{(j)}|}, \quad (6.6)$$

There is one independent θ angle for each \mathbf{P}^1 corresponding to the fact that they are all independent in homology. These θ angles will get related to chemical potentials for the D2 branes wrapping the corresponding 2-cycles.

6.1 Modularity

The S-duality of $\mathcal{N} = 4$ Yang Mills acts on our partition function as $g_s \rightarrow \frac{4\pi^2}{g_s}$. By performing this modular transformation we will be able to read off the degeneracies of the BPS bound states contributing to the entropy. First, using the definition of the Chern Simons S-matrix, we find that

$$Z = \sum_{\omega \in \mathcal{W}} (-1)^\omega \sum_{n_1, \dots, n_k \in \mathbb{Z}^N} q^{n_1^2 + \dots + n_k^2 - n_1 n_2 - \dots - n_{k-1} n_k} e^{i\theta_1 |n_1| + \dots + \theta_k |n_k|} q^{\rho n_1 + n_k \omega(\rho)} \quad (6.7)$$

Note the appearance of the intersection matrix of A_k ALE space. The fact that the Cartan matrix appears gives the k vectors $U(N)$ weight vectors n_i^a $i = 1, \dots, k$, $a = 1, \dots, N$ an alternative interpretation as N $SU(k)$ root vectors:

$$Z = \sum_{\omega \in \mathcal{W}} (-1)^\omega \prod_{a=1}^N \sum_{n_a \in \Lambda_{SU(k)}^{Root}} q^{\frac{1}{2} n_a n_a} e^{i\theta n_a} q^{(\rho + \omega(\rho))_a n_a}$$

where θ is a k -dimensional vector with entries θ_i . From the above, it is clear that Z is a product of N $SU(k)$ characters at level one. Recall that the level one characters are

$$\chi_\lambda^{(1)}(\tau, u) = \frac{\theta_\lambda^{(1)}(\tau, u)}{\eta^k(\tau)}$$

where

$$\theta_\lambda^{(1)}(\tau, u) = \sum_{n \in \Lambda_{SU(k)}^{Root}} e^{\pi i \tau (n+\lambda)^2 + 2\pi i (n+\lambda) u}$$

To be concrete, our amplitude is given as follows:

$$Z = \eta(q)^{Nk} \sum_{\omega \in \mathcal{W}} (-1)^\omega \prod_{a=1}^N \chi_0^{(1)}(\tau, u^a(\theta, \omega))$$

Here,

$$\tau = \frac{ig_s}{2\pi}, \quad u_i^a(\theta, \omega) = \frac{\theta_i}{2\pi} + \frac{ig_s}{2\pi}(\rho + \omega(\rho))^a$$

Modular transformations act on the space of level one characters as:

$$\theta_\eta^{(1)}\left(-\frac{1}{\tau}, \frac{u}{\tau}\right) = e^{-\frac{uu}{2\tau}} \sum_{\omega \in \mathcal{W}_k} (-1)^\omega \sum_{\lambda} e^{\frac{2\pi i}{k+1} \omega(\eta+\rho)(\lambda+\rho)} \theta_\lambda^{(1)}(\tau, u),$$

consequently, the dual partition function also has an expansion in terms of N level one characters. The product of N level one characters can be expanded in terms of sums of level N characters, so this is consistent with the results of H. Nakajima. The fact that the partition function is a sum over level N characters, rather than a single one is natural given that we impose different boundary conditions at the infinity of ALE space from [23].

6.2 The large N limit

In the 't Hooft large N expansion, using (4.4), we find that the partition function (6.7) can be written as follows:

$$Z_{ALE} = \sum_{P_1, \dots, P_{k+1}} (-1)^{|P_1| + \dots + |P_{k+1}|} \sum_{m_1, \dots, m_k \in \mathbf{Z}}$$

$$Z_{P_1, \dots, P_{k+1}}^+(t_1 + m_1 g_s, \dots, t_k + m_k g_s) Z_{P_1^T, \dots, P_{k+1}^T}^+(\bar{t}_1 - m_1 g_s, \dots, \bar{t}_k - m_k g_s),$$

where m 's are related to the $U(1)$ charges of representations \mathcal{R}_i as $m_i = 2\ell_i - \ell_{i-1} - \ell_{i+1}$, for $i = 1, \dots, k$ (where $\ell_0 = \ell_{k+1} = 0$). The Kahler moduli are

$$t_j = -i\theta_j, \quad j = 1, \dots, k,$$

which is what attractor mechanism predicts: Since ALE space has vanishing first Chern class, the normal bundle of its embedding in a Calabi-Yau three-fold is trivial, and consequently $\#[D_{A_k} \cap C] = 0$ where D_{A_k} is (N times) the divisor corresponding to the ALE space and C is any curve class in X .

The normalization constant α_{ALE} in (6.7) was determined by requiring the large N limit factorizes in the appropriate way.

$$\alpha_{ALE} = q^{(k+1)(\rho^2 + \frac{N}{24})} e^{\frac{N}{2g_s} \theta^T A \theta}, \quad (6.8)$$

where A is the inverse of the intersection matrix of ALE.

The chiral block in the chiral(anti-chiral) decomposition of Z_{ALE} has the form

$$\begin{aligned} Z_{P_1, \dots, P_{k+1}}^+(t_1, \dots, t_k) &= M(q)^{\frac{k+1}{2}} e^{-\frac{t_0 t^T A t}{2g_s^2} + \frac{\pi^2 (k+1) t_0}{6g_s^2}} e^{-t_0 \sum_{d=1}^{k+1} |P_d|} \times \\ &\times \sum_{R_1 \dots R_k} \hat{C}_{0R_1^T P_1} q^{\kappa_{R_1}/2} e^{-t_1 |R_1|} \hat{C}_{R_1 R_2^T P_2} q^{\kappa_{R_2}/2} e^{-t_2 |R_2|} \dots \hat{C}_{R_k 0 P_{k+1}}. \end{aligned}$$

where

$$t_0 = \frac{1}{2} N g_s. \tag{6.9}$$

We see that the trivial chiral block $Z_{0,\dots,0}^+(t_1, \dots, t_k)$ is exactly the topological string partition function on ALE, in agreement with the conjecture of [1]. Moreover, the higher chiral blocks correspond to having $k + 1$ sets of topological “ghost“ branes in the \mathbf{C} direction over the north and the south poles of the \mathbf{P}^1 's. The associated moduli, i.e. the size of the holomorphic disks ending on the topological ghost branes is also fixed by the attractor mechanism, to be $\#(D_{A_k} \cap C_{disk}) = N$. This gives exactly (6.9) as the value of the corresponding Kahler moduli t_0 , in agreement with the conjecture. As we discussed in section 4, in the closed string language, these are the non-normalizable modes in the topological string on X . The classical piece of the topological string amplitude

$$\frac{1}{2g_s^2} t_0 t^T A t \tag{6.10}$$

deserves a comment. Because $X = A_k \times \mathbf{C}$, taking only the compact cohomology the triple intersection numbers would unambiguously vanish. The non-vanishing triple intersection numbers can be gotten only by a suitable regularization of the \mathbf{C} factor. This *was* already regularized, in terms of the Kahler modulus t_0 of the non-normalizable modes – which exactly give the measure of the size of the disk, i.e. \mathbf{C} , making (6.10) a natural answer.¹²

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A. Conventions and useful formulas

The S matrix is given by

$$S_{\mathcal{R}\mathcal{Q}}(N, g_s) = \sum_{w \in S_N} (-)^w q^{-w(\mathcal{R} + \rho_N) \cdot (\mathcal{Q} + \rho_N)}$$

where $q = \exp(-g_s)$, and $\rho_N^a = \frac{N-2a+1}{2}$, for $a = 1, \dots, N$. Note that while the expression for $S_{\mathcal{R}\mathcal{Q}}$ looks like that for the S-matrix of the $U(N)$ WZW model, unlike in

¹²What is less natural is the appearance of the *inverse* intersection matrix of ALE. However, one has to remember that this is a non-compact Calabi-Yau, where intersection numbers are inherently ambiguous.

WZW case, g_s is not quantized. Using Weyl denominator formula $Tr_{\mathcal{R}}x = \prod_{i < j} (x_i - x_j) \sum_{w \in S_N} (-)^w x^{w(\mathcal{R} + \rho_N)}$, the S -matrix can also be written in terms of Schur functions $s_{\mathcal{R}}(x_1, \dots, x_N) = Tr_{\mathcal{R}}x$ of N variables.

$$S_{\mathcal{R}\mathcal{Q}}/S_{00}(g_s, N) = s_{\mathcal{R}}(q^{-\rho_N - \mathcal{Q}})s_{\mathcal{Q}}(q^{-\rho_N}).$$

Above, x is an N by N matrix with eigenvalues $x_i, i = 1, \dots, N$, as

The S matrix has following important properties:

$$S_{\bar{\mathcal{R}}\mathcal{Q}}(N, g_s) = S_{\mathcal{R}\mathcal{Q}}(N, -g_s) = S_{\mathcal{R}\mathcal{Q}}^{-1}(N, g_s)$$

The first follows since (up to a sign that is $+1$ if N is odd and -1 if N is even), $\bar{\mathcal{Q}} + \rho_N = -\omega_N(\mathcal{Q} + \rho_N)$ where ω_N is the permutation that maps $a \rightarrow N - a + 1$ for $a = 1, \dots, N$. The second is easily seen by computing

$$\begin{aligned} \sum_{\mathcal{P}} S_{\mathcal{R}\mathcal{P}}(N, -g_s) S_{\mathcal{P}\mathcal{Q}}(N, g_s) &= \sum_{w \in S_N} (-)^w \sum_{n \in \mathbf{Z}^N} q^{w(\rho_N + \mathcal{R}) \cdot n} q^{-n \cdot (\rho_N + \mathcal{Q})} \\ &= \sum_{w \in S_N} (-)^w \delta^{(N)}(w(\rho_N + \mathcal{R}) - (\rho_N + \mathcal{Q})) = \delta_{\mathcal{R}\mathcal{Q}}. \end{aligned}$$

where we absorbed one sum over the Weyl group into the unordered vector, n . Note that $(\rho_N + \mathcal{R})^a$ and $(\rho_N + \mathcal{Q})^a$ are decreasing in a , so the delta function can only be satisfied when $w = 1$.

The large N limit of the S matrix for coupled representations $\mathcal{R} = R_+ \bar{R}_-[\ell_R]$, $\mathcal{Q} = Q_+ \bar{Q}_-[\ell_Q]$ is given in (4.4) in terms of the topological vertex amplitude

$$\hat{C}_{RQP}(q) = C_{RQP}(q^{-1}), \quad C_{RQP}(q) = q^{\frac{\kappa R}{2}} s_P(q^\rho) \sum_{\eta} s_{R^t/\eta}(q^{P+\rho}) s_{Q/\eta}(q^{P^t+\rho})$$

This has cyclic symmetry $\hat{C}_{PQR} = \hat{C}_{QRP}$, and using the properties of the Schur functions under $q \rightarrow q^{-1}$: $s_R(q^{Q+\rho}) = (-1)^{|R|} s_{R^T}(q^{-Q^T-\rho})$ also a symmetry under inversion: $\hat{C}_{RQP}(q^{-1}) = (-1)^{|R|+|Q|+|P|} \hat{C}_{R^t Q^t P^t}(q)$. The leading piece of S in the large N limit is significantly simpler than (4.4). Since $\hat{C}_{0RQ}(q) = (-1)^{|R|+|Q|} W_{R^T Q}(q) q^{-\frac{1}{2}\kappa Q}$ we have:

$$\begin{aligned} S_{\mathcal{R}\mathcal{Q}}(g_s, N) &= (-1)^{|R_+|+|Q_+|+|R_-|+|Q_-|} q^{-N\ell_R\ell_Q} q^{-\ell_R(|Q_+|-|Q_-|)} q^{-\ell_Q(|R_+|-|R_-|)} \\ &\quad W_{R_+Q_+}(q) W_{R_-Q_-}(q) q^{-\frac{N}{2}(|R_+|+|R_-|+|Q_+|+|Q_-|)} \end{aligned}$$

where

$$W_{RQ}(q) = s_R(q^{\rho+Q}) s_Q(q^\rho)$$

where $\rho = -a + \frac{1}{2}$, for $a = 1, \dots, \infty$.

B. Quantum Yang-Mills amplitudes with observable insertions

Consider the $U(N)$ q -deformed YM path integral on the cap. As shown in [3] this is given by

$$Z_{\text{qYM}}(C)(U) = \sum_{\mathcal{R}} S_{0\mathcal{R}} Tr_{\mathcal{R}} U.$$

The Fourier transform to the Φ basis is given by the following path integral over the boundary of the disk,

$$Z_{\text{qYM}}(C)(U) = \int d_H \Phi e^{\frac{1}{g_s} \text{Tr} \Phi \oint A} Z_{\text{qYM}}(C)(\Phi).$$

Since the qYM path integral localizes to configurations where Φ is covariantly constant, so in particular Φ and A commute, integrating over the angles gives¹³

$$Z_{\text{qYM}}(C)(\vec{u}) = \int \prod_i d\phi_i \frac{\Delta_H(\phi)}{\Delta_H(u)} e^{\frac{1}{g_s} \sum_i \vec{\phi} \cdot \vec{u}} Z_{\text{qYM}}(C)(\vec{\phi}),$$

where we defined a hermitian matrix u by $U = e^{iu}$, and

$$\Delta_H(\phi) = \prod_{1 \leq i < j \leq N} 2 \sin[(\phi_i - \phi_j)/2] = \prod_{\alpha > 0} 2 \sin(\vec{\alpha} \cdot \vec{\phi}).$$

comes from the hermitian matrix measure over $\vec{\phi}$ by adding images under $\vec{\phi} \rightarrow \vec{\phi} + 2\pi\vec{n}$, to take into account the periodicity of Φ .

Now, in the Φ basis, the path integral on the disk with insertion of $\text{Tr}_{\mathcal{Q}} e^{i\Phi}$ is simply given by:

$$Z(C, \text{Tr}_{\mathcal{Q}} e^{i\Phi})(\Phi) = \text{Tr}_{\mathcal{Q}} e^{i\Phi}$$

since Φ is a multiplication operator in this basis. Transforming this to U -basis, we use

$$\text{Tr}_{\mathcal{Q}} e^{i\Phi} := \chi_{\mathcal{Q}}(\vec{\phi}) = \frac{\sum_{\omega \in S_N} (-1)^\omega e^{i\omega(\vec{\mathcal{Q}} + \vec{\rho}) \cdot \vec{\phi}}}{\sum_{\omega \in S_N} (-1)^\omega e^{i\omega(\vec{\rho}) \cdot \vec{\phi}}},$$

where S_N is the Weyl group and $\vec{\rho}$ is the Weyl vector. We also use the Weyl denominator formula

$$\prod_{\alpha > 0} \sin(\vec{\alpha} \cdot \vec{\phi}) = \sum_{\omega \in S_N} (-1)^\omega e^{i\omega(\vec{\rho}) \cdot \vec{\phi}}.$$

Plugging this into the integral, and performing a sum over the weight lattice we get

$$Z(C, \text{Tr}_{\mathcal{Q}} e^{i\Phi})(U) = \frac{1}{\Delta_H(u)} \sum_{\omega \in W} (-1)^\omega \delta(\vec{u} + ig_s \omega(\vec{\rho} + \vec{\mathcal{Q}}))$$

We can extract the coefficient of this in front of $\text{Tr}_{\mathcal{R}} U$ by computing an integral

$$\int dU Z(C, \text{Tr}_{\mathcal{Q}} e^{i\Phi})(U) \text{Tr}_{\mathcal{R}} U^{-1}$$

which easily gives

$$Z(C, \text{Tr}_{\mathcal{Q}} e^{i\Phi})(U) = \sum_{\mathcal{R}} S_{\mathcal{R}\mathcal{Q}}(g_s, N) \text{Tr}_{\mathcal{R}} U.$$

¹³There was an error in [3] where the denominator $1/\Delta_H(u)$ was dropped. In that case this only affected the definition of the wave function (whether one absorbs the determinant $\Delta_H(\phi)$ into the wave function of ϕ or not), but here we need the correct expression. This normalization follows from [24] where the matrix model for a pair of commuting matrices with haar measure was first discussed.

where

$$S_{\mathcal{R}\bar{\mathcal{Q}}}(g_s, N) = \sum_{\omega} q^{\omega(\mathcal{Q}+\rho)\cdot(\mathcal{R}+\rho)}$$

in terms of $q = e^{-g_s}$.

Another expectation value we need is of

$$Z(C, Tr_{\mathcal{Q}} e^{i\Phi - in \oint A})(U)$$

We can compute this by replacing Φ by $\Phi' = \Phi - n \oint A$ everywhere. The only difference is that we must now transform from $\Phi - n \oint A$ basis (with $\oint A$ as a momentum) where the computations are simple to $\oint A$ basis with Φ as a momentum, and this is done by

$$Z_{2dYM}(C)(U) = \int d\Phi' e^{\frac{1}{g_s} \text{Tr} \Phi' u + \frac{n}{2g_s} \text{Tr} u^2} Z_{2dYM}(C)(\Phi').$$

This gives

$$Z(C, Tr_{\mathcal{Q}} e^{i\Phi - in \oint A})(U) = \sum_{\mathcal{R}} q^{\frac{n}{2} C_2(\mathcal{Q})} S_{\bar{\mathcal{Q}}\mathcal{R}} Tr_{\mathcal{R}} U$$

C. Modular transformations

C.1 The vertex amplitude

Consider the vertex amplitude corresponding to intersecting D4 branes:

$$Z(U, V) = \sum_{\mathcal{R} \in U(N), \mathcal{Q} \in U(M)} V_{\mathcal{R}\mathcal{Q}}(N, M) Tr_{\mathcal{R}} U Tr_{\mathcal{Q}} V$$

where

$$V_{\mathcal{R}\mathcal{Q}} = \sum_{\mathcal{P} \in U(M)} q^{\frac{C_2^{(M)}(\mathcal{P})}{2}} S_{\mathcal{R}\mathcal{P}}(g_s, N) S_{\mathcal{P}\mathcal{Q}}(-g_s, M)$$

Using the definition (3.8) of $S_{\mathcal{R}\mathcal{Q}}$ and the Weyl-denominator form of the $U(N)$ characters $Z(U, V)$ becomes:

$$Z(U, V) = \frac{1}{\Delta_H(u)\Delta_H(v)} \sum_{w_1, w_3 \in S_N} \sum_{w'_3, w_2 \in S_M} (-)^{w_1+w_2+w_3+w'_3} q^{\frac{\|P+\rho_M\|^2}{2}} \\ q^{(\mathcal{P}+\rho_M)\cdot w'_3(\mathcal{Q}+\rho_M)} q^{-(\mathcal{P}+\rho_N)\cdot w_3(\mathcal{R}+\rho_N)} e^{i w_1(\mathcal{R}+\rho_N)\cdot u} e^{(\mathcal{Q}+\rho_M)\cdot w_2(iv)}$$

We can trade the sums over the Weyl groups, for sums over the full weight lattices: Put

$$w_2 = w_Q^{-1}, \quad w'_3 = w_P^{-1} w_Q,$$

this defines elements $w_P, w_Q \in S_M$ uniquely given w_2, w'_3 . Then, we can always find an element $w_R \in S_N$ such that

$$w_3 = w_P^{-1} w_R,$$

for a given w_3 , by simply viewing w_P as an element of S_N acting on first M entries of any N dimensional vector, leaving the others fixed. Finally, find an $w \in S_N$ such that

$$w_1 = w^{-1}w_R,$$

Note now that

$$\omega_P(P + \rho_N) = \omega_P(P + \rho_M) + \rho_N - \rho_M$$

since ω_P acts only on first M entries of a vector and the first M entries of $\rho_N - \rho_M$ are all equal, hence invariant under ω_P . Using this and the fact that now only permutations w are counted with alternating signs, we can combine the sums over the weyl-groups with the sums over the lattices to write:

$$Z(U, V) = \Delta_H(u)^{-1} \Delta_H(v)^{-1} \sum_{w \in S_N} (-)^w \sum_{m, p \in \mathbf{Z}^M; n \in \mathbf{Z}^N} q^{\frac{p^2}{2}} q^{p \cdot m} q^{-(p + \rho_N - \rho_M) \cdot n} e^{in \cdot w(u)} e^{im \cdot v}$$

Now split $n = (n', n'')$ where n' is the first M entries in n , n'' the remaining $N - M$, and similarly put $\rho_N - \rho_M = (\rho', \rho'')$, where we have treated ρ_M as N dimensional vector first M entries of which is the standard Weyl vector of $U(M)$, the remaining being zero, and $u = (u', u'')$. If one in addition defines $m' = m - n'$ above becomes

$$\begin{aligned} Z(U, V) = \theta^M(q) \Delta_H(u)^{-1} \Delta_H(v)^{-1} \sum_{w \in S_N} (-)^w \sum_{m' \in \mathbf{Z}^M} q^{-\frac{(m')^2}{2}} e^{im' \cdot v} \\ \sum_{n' \in \mathbf{Z}^M} \sum_{n'' \in \mathbf{Z}^{N-M}} q^{-\rho' \cdot n' - \rho'' \cdot n''} e^{n' \cdot (w(iu') + iv) + n'' \cdot w(iu'')} \end{aligned} \quad (\text{C.1})$$

where $\theta(q) = \sum_{n \in \mathbf{Z}} q^{\frac{n^2}{2}}$ is the usual theta function. We write n again as an N -dimensional vector $(n', n'') = n$ to get our final expression

$$Z(U, V) = \theta(q)^M \Delta_H(u)^{-1} \Delta_H(v)^{-1} \sum_{m' \in \mathbf{Z}^M} q^{-\frac{(m')^2}{2}} e^{im' \cdot v} \sum_{w \in S_N} (-)^w \delta(iv + w(iu) + (\rho_N - \rho_M)g_s)$$

where v, ρ_M are regarded as N dimensional vectors $(v, 0^{N-M}), (\rho_M, 0^{N-M})$.

D. Large N limit of the vertex amplitude

Here we find the large N, M limit of the interaction

$$V_{\mathcal{RQ}} = \sum_{\mathcal{P}} S_{\mathcal{RP}}(N, g_s) q^{\frac{c_2^{(M)}(\mathcal{P})}{2}} S_{\overline{\mathcal{PQ}}}(M, g_s)$$

(we've dropped an overall factor). Using $TS^{-1} = \theta(q)^M S^{-1}T^{-1}S^{-1}T^{-1}$ in the $U(M)$ factor, this can be done by computing first the large N, M limit of

$$\sum_{\mathcal{P}} S_{\mathcal{RP}}(N, g_s) S_{\overline{\mathcal{PA}}}(M, g_s)$$

and then using large M limit of $(TST)_{\mathcal{A}\mathcal{Q}}^{-1}$ to get the full amplitude. In general, either version of the problem is very difficult and at present unsolved. Things simplify significantly if we *turn off* the $U(1)$ charges all together. This means we will effectively compute the $SU(N)$ rather than $U(N)$ version of interaction. It will turn out that the crucial features that one expects from the amplitudes *assuming* the conjecture holds, are unaffected by this. In this case, the representations \mathcal{R} are effectively labeled by Young tableaux's.

From the free fermion description of the YM amplitudes it follows easily [25] that:

$$\begin{aligned} & \sum_{\mathcal{P} \in U(M)} S_{\mathcal{R}\mathcal{P}}(g_s, N) S_{\bar{\mathcal{P}}\mathcal{A}}(g_s, M) \rightarrow \alpha_N^{-1}(q) \alpha_M^{-1}(q) S_{0(R_+ \bar{R}_-)}(g_s, N) S_{0(A_+ \bar{A}_-)}(g_s, M) \\ & \times \prod_{i,j=1}^{\infty} \frac{[\frac{1}{2}N - \frac{1}{2}M + j - i]}{[R_i^+ - A_j^+ + \frac{1}{2}N - \frac{1}{2}M + j - i]} \frac{[\frac{1}{2}N - \frac{1}{2}M + j - i]}{[R_i^- - A_j^- + \frac{1}{2}N - \frac{1}{2}M + j - i]} \\ & \times \prod_{i,j=1}^{\infty} \frac{[\frac{1}{2}N + \frac{1}{2}M - j - i + 1]}{[R_i^+ + A_j^- + \frac{1}{2}N + \frac{1}{2}M - j - i + 1]} \frac{[\frac{1}{2}N + \frac{1}{2}M - j - i + 1]}{[R_i^- + A_j^+ + \frac{1}{2}N + \frac{1}{2}M - j - i + 1]} \quad (\text{D.1}) \end{aligned}$$

where the arrow indicates taking large N, M limit and where $\alpha_N(q) = q^{-(\rho_N^2 + \frac{N}{24})} M(q) \times \eta^M(q)$, and similarly for α_M with M, N exchanged.

For simplicity, we will be interested only in the leading chiral block of the amplitude which determines the Calabi-Yau manifold that the YM theory describes in the large N limit, and neglects the excitations of non-normalizable modes. In this limit, the piece $S_{0(R_+ \bar{R}_-)}(g_s) S_{0(A_+ \bar{A}_-)}(g_s)$ gives

$$\alpha_M(q) \alpha_N(q) W_{A_+^T 0}(q) W_{A_-^T 0}(q) W_{R_+^T 0}(q) W_{R_-^T 0}(q) q^{-\frac{M(|A_+| + |A_-|)}{2}} q^{-\frac{N(|R_+| + |R_-|)}{2}}$$

where $W_{RP}(q) = s_R(q^\rho) s_P(q^{R+\rho})$, and moreover $W_{R0}(q) = (-)^{|R|} q^{k_R/2} W_{R^T 0}(q)$. Of the infinite product terms, in the leading chiral block limit only the second row in (D.1) contributes. This is because the interactions between the chiral and anti-chiral part of the amplitude are suppressed in this limit. Using

$$\prod_{i,j} \frac{1}{x_i - y_j} = \prod_i x_i^{-1} \sum_R s_R(x^{-1}) s_R(y)$$

we get

$$const. \times \sum_{P_+, P_-} s_{P_+}(q^{R_+ + \rho}) s_{P_+}(q^{-(A_+ + \rho)}) s_{P_-}(q^{R_- + \rho}) s_{P_-}(q^{-(A_- + \rho)}) q^{(|P_+| + |P_-|) \frac{N-M}{2}}$$

The constant comes from regularizing the infinite products (see [25] for details) and can be determined by computing the leading large N, M scaling

$$\begin{aligned} & \prod_{(i,j) \in A_+} \frac{[\frac{1}{2}N - \frac{1}{2}M - j + i]}{[-\frac{1}{2}N - \frac{1}{2}M - j + i]} \prod_{(i,j) \in A_-} \frac{[\frac{1}{2}N - \frac{1}{2}M - j + i]}{[-\frac{1}{2}N - \frac{1}{2}M - j + i]} \\ & \prod_{(i,j) \in R_+} \frac{[\frac{1}{2}N - \frac{1}{2}M + j - i]}{[\frac{1}{2}N + \frac{1}{2}M + j - i]} \prod_{(i,j) \in R_-} \frac{[\frac{1}{2}N - \frac{1}{2}M + j - i]}{[\frac{1}{2}N + \frac{1}{2}M + j - i]} \sim q^{\frac{\kappa_{A_+} + \kappa_{A_-}}{2}} q^{\frac{M(|R_+| + |R_-| + |A_+| + |A_-|)}{2}} \end{aligned}$$

where i goes over the rows and j over the columns. All together, this gives

$$\begin{aligned} \sum_{\mathcal{P} \in U(M)} S_{\mathcal{RP}}(g_s, N) S_{\overline{\mathcal{P}}\mathcal{A}}(g_s, M) &\rightarrow (-)^{|R_+|+|R_-|} q^{-\frac{N-M}{2}(|R_+|+|R_-|)} q^{-\frac{\kappa_{R_+}+\kappa_{R_-}}{2}} q^{\frac{\kappa_{A_+}+\kappa_{A_-}}{2}} \\ &\sum_{P_+} (-)^{|P_+|} W_{R_+P_+}(q) W_{P_+^T A_+^T}(q) q^{\frac{N-M}{2}|P_+|} \\ &\sum_{P_-} (-)^{|P_-|} W_{R_-P_-}(q) W_{P_-^T A_-^T}(q) q^{\frac{N-M}{2}|P_-|} \end{aligned}$$

Next, recall that (see appendix A) the large M limit (more precisely, the leading chiral block) of $(TST)^{-1}$ is

$$(T^{-1}S^{-1}T^{-1})_{A\mathcal{Q}} = \alpha_M(q^{-1}) W_{A_+Q_+}(q) W_{A_-Q_-}(q) q^{-\frac{\kappa_{A_+}+\kappa_{A_-}+\kappa_{Q_+}+\kappa_{Q_-}}{2}}$$

To compute our final expression, we need to sum:

$$\begin{aligned} \sum_{\mathcal{P}} S_{\mathcal{RP}}(N, g_s) q^{\frac{C_2^{(M)}(\mathcal{P})}{2}} S_{\overline{\mathcal{P}}\mathcal{Q}}(M, g_s) &\rightarrow \alpha_M(q^{-1}) (-)^{|R_+|+|R_-|} q^{-\frac{N-M}{2}(|R_+|+|R_-|)} \\ &q^{-\frac{\kappa_{R_+}+\kappa_{Q_+}}{2}} \sum_{P_+, A_+} (-)^{|P_+|} W_{R_+P_+}(q) W_{P_+^T A_+^T}(q) W_{A_+Q_+}(q) q^{\frac{N-M}{2}|P_+|} \\ &q^{-\frac{\kappa_{R_-}+\kappa_{Q_-}}{2}} \sum_{P_-, A_-} (-)^{|P_-|} W_{R_-P_-}(q) W_{P_-^T A_-^T}(q) W_{A_-Q_-}(q) q^{\frac{N-M}{2}|P_-|}. \end{aligned}$$

Note that this contains an ill-defined expression

$$\sum_{A_+} W_{P_+^T A_+^T}(q) W_{Q_+A_+}(q) \sum_{A_-} W_{P_-^T A_-^T}(q) W_{Q_-A_-}(q) \quad (\text{D.2})$$

The physical interpretation of a finite version of this amplitude, with insertions of $e^{-t|A_+|}$ and $e^{-\bar{t}|A_-|}$ also suggests how to define (D.2). Namely, the finite amplitude is the topological string amplitude (more precisely, two copies of it) on $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$ with D-branes as in the figure, where the size of the \mathbf{P}^1 is t .

In the limit $t \rightarrow 0$ the \mathbf{P}^1 shrinks to zero size, and one can undergo a conifold transition, to a small S^3 of size ϵ . In this case, the only holomorphic maps correspond to those with $P_+ = Q_+$, so that

$$\sum_{A_+} W_{P_+^T A_+^T}(q) W_{Q_+A_+}(q) = \delta(P_+ - Q_+),$$

and similarly for the anti-chiral piece, which is independent of ϵ , as this is a complex structure parameter. Our final result is:

$$\begin{aligned} \sum_{\mathcal{P}} S_{\mathcal{RP}}(N, g_s) q^{\frac{C_2^{(M)}(\mathcal{P})}{2}} S_{\overline{\mathcal{P}}\mathcal{Q}}(M, g_s) &\rightarrow \alpha_M(q^{-1}) \theta^M(q) (-)^{|R_+|+|R_-|+|Q_+|+|Q_-|} \\ &q^{-\frac{N-M}{2}(|R_+|+|R_-|)} q^{\frac{N-M}{2}(|Q_+|+|Q_-|)} q^{-\frac{\kappa_{R_+}+\kappa_{R_-}}{2}} q^{-\frac{\kappa_{Q_+}+\kappa_{Q_-}}{2}} W_{R_+Q_+}(q) W_{R_-Q_-}(q) \quad (\text{D.3}) \end{aligned}$$

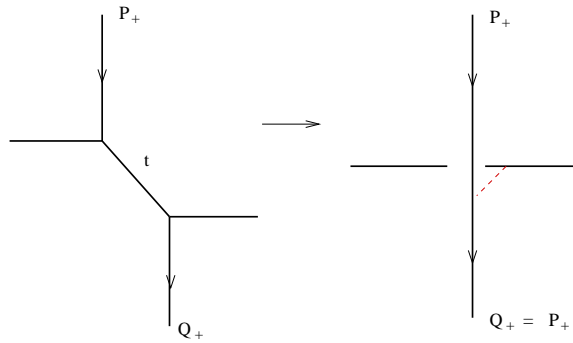


Figure 6: The figure on the left corresponds to $O(-1) \oplus O(-1) \rightarrow \mathbf{P}^1$ with \mathbf{P}^1 of size t with two stacks of lagrangian D-branes. The representations P_+ and Q_+ label the boundary conditions on open string maps. When $t = 0$ the Calabi-Yau is singular, but can be desingularized by growing a small S^3 . The singular topological string amplitudes can be regulated correspondingly, and with this regulator, they vanish unless $P_+ = Q_+$. See [26] for more details.

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